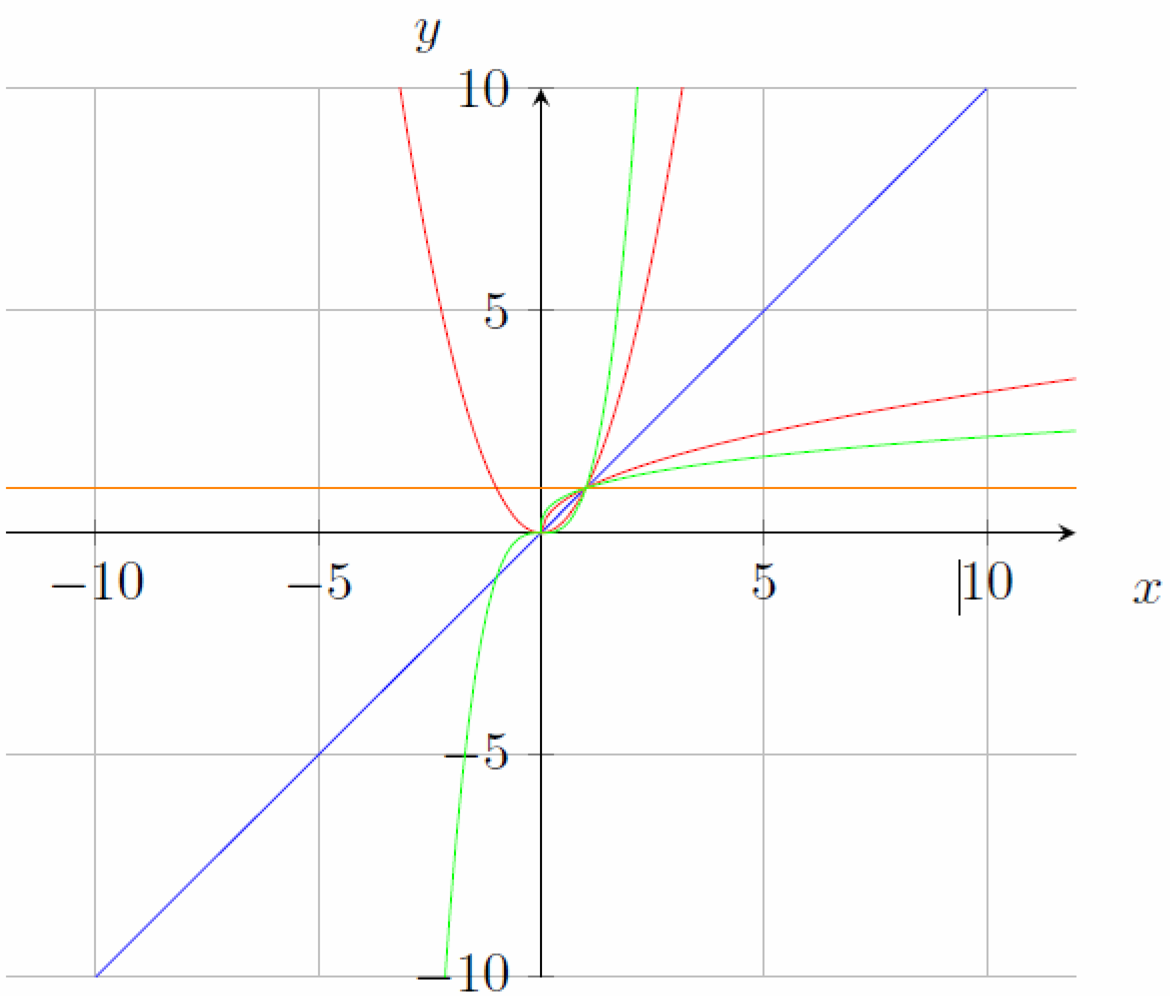
Mathematics for Architects II - Analysis of functions



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Pécs

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Mathematics for Architects II - Analysis of functions

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Chapter 1

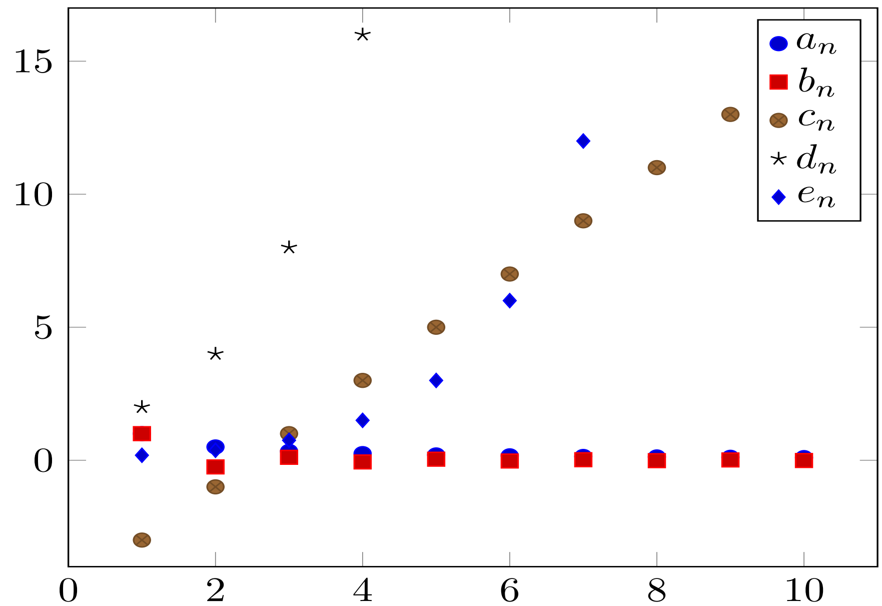
Sequences

1.1 Definition

A *sequence of numbers* is a function mapping natural numbers to real numbers, that is, a sequence of numbers indexed by natural numbers. Notation: $\{a_n\}, \ n \in \mathbb{N}$. n is called the index, showing the place of the given value in the sequence. A few examples:

1) $a_n = \frac{1}{n}, \quad n \in \mathbb{N}^+$, that is: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots$
2) $b_n = (-1)^{n+1} \frac{1}{n^2}, \quad n \in \mathbb{N}^+$, that is: $\frac{1}{1^2}, -\frac{1}{2^2}, \frac{1}{3^2}, -\frac{1}{4^2}, \ldots$
3) $c_n = 2n - 5, \quad n \in \mathbb{N}$
4) $d_n = 2^n, \quad n \in \mathbb{N}$
5) $e_n = 3 \cdot 2^{n-5}, \quad n \in \mathbb{N}$

Let's plot the values of the above sequences in a coordinate system, to see the tendencies.



1) We can see that the values are closer and closer to zero, as n is getting bigger.

2) The values of the sequence have an alternating sign, otherwise the tendency is the same: the values are getting closer and closer to zero, as n gets bigger. The tendency is a bit stronger (faster) than in the previous case.

3) This is an arithmetic sequence, its values follow a straight line (it is a linear function).

4) This is a geometric sequence, each value is twice as much as the previous value (it is an exponential function).

5) This is again a geometric sequence, with each value being twice as much as the previous value (it is also an exponential function).

1.2 Properties of sequences

Now you have an idea about the notion of sequences. Let's write down a few important properties they can have.

1.2.1 Boundedness

A sequence is *bounded from above* if its values never exceed a fixed boundary. For example the first and second sequences are bounded from above, all their values are smaller or equal to 1 (and 1 is the least upper bound). They are also *bounded from below*: none of their values are lower than an other fixed boundary: all their elements are greater than -1. Actually, the greatest lower bound for the second sequence is $-\frac{1}{2}$, and for the first sequence the greatest lower bound is zero. If we only say they are *bounded* it means they are bounded both from above and below. This is true for the first and second sequence. The 3rd, 4th and 5th sequences are not bounded from above, but they are bounded from below.

The formal definition is: we say the sequence a_n is bounded from above if there exists a $K \in \mathbb{R}$ such that for all n, $a_n \leq K$. Similarly, the sequence a_n is bounded from below if there exists a $k \in \mathbb{R}$ such that for all n, $a_n \geq k$. The sequence a_n is bounded if it is bounded from above and also bounded from below.

1.2.2 Having a limit

We say about the first two sequences that they tend to zero, because their values are getting closer and closer to zero. An other word for the same phenomenon is *convergence*: if they tend to a finite real number A, we say the sequence is convergent, it is converging to A.

The formal definition is: we say the sequence a_n converges to A if for any $\epsilon > 0$ there exists an index $N \in \mathbb{N}$ (depending on ϵ) such that for all $n \geq N$, a_n is very close to A: a_n falls in the $(A - \epsilon, A + \epsilon)$ interval, or equivalently, $|a_n - A| < \epsilon$. We write $\lim_{n\to\infty} a_n = 0$.

Equivalent definition: for any fixed $\epsilon > 0$, the sequence only has finitely many elements out of the interval $(A - \epsilon, A + \epsilon)$.

From the above five examples, only the first two sequences are convergent. But the 3rd, 4th and 5th sequences also have a tendency: their values are getting bigger and bigger, growing above any boundary. We can say that they tend to infinity. They are not called convergent, but we still can say they have a *limit*. Their limit is $+\infty$. We write $\lim_{n\to\infty} c_n = \infty$. In the case of the first two sequences, tending to zero, we can also say their limit is zero: e.g. $\lim_{n\to\infty} a_n = 0$.

If we take the sum of the elements of the fist two sequences for each n, the new sequence is still converging to zero, according to the definition. It is true in general: if a_n and b_n are both convergent, then $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.

If we take the negative of the elements of the 3rd, 4th or 5th sequence for each n, the new sequence will tend to $-\infty$: e.g. $\lim_{n\to\infty}(-c_n) = -\infty$.

We say the sequence is *divergent* if it is not convergent, even if its limit is ∞ or $-\infty$.

1.2.3 Accumulation points

A *subsequence* consists of some of the elements of the original sequence, possibly not all (re-indexed by the set of natural numbers).

The sequence has an *accumulation point* $A \in \mathbb{R}$, if it has a subsequence converging to A, or equivalently, there are infinitely many elements of the sequence in any neighbourhood $(A - \epsilon, A + \epsilon)$ of A.

An other example for a divergent sequence is, when it has more than one accumulation points.

Theorem: All convergent sequences are bounded.

Proof. According to the second (equivalent) definition of convergence, for any fixed $\epsilon > 0$, the sequence only has finitely many elements out of the interval $(A-\epsilon, A+\epsilon)$. Let $\epsilon = 1$. There are only finitely many elements out of the neighbourhood $(A-1, A+1)$. Let k denote the minimum of these outstanding values, and let K denote their maximum. $k' = min\{A-1, k\}$ and $K' = max\{A+1, K\}$ will be a good lower and upper bound, respectively.

Remark. The reverse of the statement is not true: the sequence $a_n = (-1)^n, \ \ n \in \mathbb{N}$ is bounded, but not convergent.

Theorem: Any subsequence of a convergent sequence is also convergent, and its limit is the same as the limit of the original sequence.

1.2.4 Monotonic sequences

A sequence is *monotonically increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$. If $a_n < a_{n+1}$ for all $n \in \mathbb{N}$, then we say the sequence is *strictlyd monotonically increasing*. The definition of a (strictly) monotonically decreasing sequence is similar.

Theorem: If a sequence is bounded and it is monotonically increasing or monotonically decreasing, then it is convergent.

1.3 Finding the limit of a sequence

There are some important sequences whose limit you must know.

a) $a_n = c \ (c \in \mathbb{R})$	$\lim_{n\to\infty} c = c$		provable by definition
b) $a_n = \frac{1}{n}$	$\lim_{n\to\infty} \frac{1}{n} = 0$		provable by definition
c) $a_n = q^n \ (q \in \mathbb{R})$	$\lim_{n\to\infty} q^n = \infty$	if $q > 1$	divergent
	$\lim_{n\to\infty} q^n = 1$	if $q = 1$	convergent
	$\lim_{n\to\infty} q^n = 0$	if $\lvert q\rvert < 1$	convergent
	q^n is divergent	if $q = -1$	divergent, no limit
	q^n is divergent	if $q < -1$	divergent, no limit
d) $a_n = \sqrt[n]{a} \ (a > 0)$	$\lim_{n\to\infty} \sqrt[n]{a} = 1$		convergent
e) $a_n = \left(1+\frac{1}{n}\right)^n$	$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$		convergent

Theorem: The sequence $a_n = \left(1+\frac{1}{n}\right)^n$ is convergent.
Proof. First we prove that the sequence is (strictly) monotonically increasing. A well-known inequality holds between the geometric and arithmetic means:

$$\sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n} \leq \frac{x_1 + x_2 + \ldots + x_n}{n},$$

where $x_1, x_2, \ldots, x_n$ are non-negative real numbers. For $x_1 = 1+\frac{1}{n}, \ldots, x_n = 1+\frac{1}{n}$, but $x_{n+1} = 1$, this inequality implies

$$\sqrt[n+1]{\left(1+\frac{1}{n}\right)^n \cdot 1} < \frac{n \cdot \left(1+\frac{1}{n}\right)+1}{n+1} = \frac{n+1+1}{n+1}$$

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}$$

Using the same inequality, we can also prove that the sequence $b_n = \left(1+\frac{1}{n}\right)^{n+1}$ is (strictly) monotonically decreasing. Now let $x_1 = \frac{1}{1+\frac{1}{n}}, \ldots, x_{n+1} = \frac{1}{1+\frac{1}{n}}$, but $x_{n+2} = 1$. Then

$$\sqrt[n+2]{\left(\frac{1}{1+\frac{1}{n}}\right)^{n+1} \cdot 1} < \frac{(n+1)\left(\frac{1}{1+\frac{1}{n}}\right)+1}{n+2} = \frac{n+1}{n+2} = \frac{1}{1+\frac{1}{n+1}}$$

$$\left(\frac{1}{1+\frac{1}{n}}\right)^{n+1} < \left(\frac{1}{1+\frac{1}{n+1}}\right)^{n+2},$$

so

$$\left(1+\frac{1}{n}\right)^{n+1} > \left(1+\frac{1}{n+1}\right)^{n+2},$$

as stated. Now we can see that

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1} < \ldots < \left(1+\frac{1}{n+1}\right)^{n+2} < \left(1+\frac{1}{n}\right)^{n+1}.$$

This means both a_n and b_n are bounded and monotonic (they give bounds for each other), so they are both convergent, with the same limit. Their limit is an important (irrational) number, called e after a great mathematician, Euler: $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \approx 2.71828\ldots$.

When we speak about *the sum, difference, product or quotient of two sequences*, then we mean the sequence whose nth element is the sum, difference, product or quotient of the nth elements of the original two sequences, respectively.

Theorem: Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences, and let A and B be their limits, respectively ($A, B \in \mathbb{R}$). Then their sum, difference, product and quotient are also convergent:
$\lim_{n\to\infty}(a_n \pm b_n) = A \pm B$
$\lim_{n\to\infty}(a_n b_n) = AB$
$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{A}{B}$, if $B \neq 0$ and b_n is not zero for any $n \in \mathbb{N}$.

If not both of the sequences are convergent, but at least one of them tends to ∞ or $-\infty$, in some cases we still know what is the limit of their sum, difference, product or quotient, but not in all cases. A few examples of what we *know*:
$A + \infty = \infty$ if $A \in \mathbb{R}$
$\infty + \infty = \infty$
$\infty \cdot \infty = \infty$
$\frac{A}{\infty} = \frac{A}{-\infty} = 0$ if $A \in \mathbb{R}$
$\frac{\infty}{A} = \infty$ if $A > 0,\ A \in \mathbb{R}$
A few examples of what we *do not know*:
$\infty - \infty =?$ (any result is possible)
$\infty \cdot 0 =?$ (any result is possible)
$\frac{\infty}{\infty} =?$ (any result is possible)
$1^\infty =?$ (any positive result is possible)
A few examples for the $\infty - \infty$ case: $a_n = n$ and $b_n = n^2$ ($n \in \mathbb{N}$) both tend to infinity. $lim_{n\to\infty}(a_n - b_n) = -\infty$, but $lim_{n\to\infty}(b_n - a_n) = \infty$. Also, $c_n = n + 4$ ($n \in \mathbb{N}$) tends to infinity, but $lim_{n\to\infty}(c_n - a_n) = 4$. Etc.
For the type 1^∞, $a_n = \left(1 + \frac{1}{n}\right)^n$ was a non-trivial example. Based on the theorem previously proved, it is not difficult to see that

$$\lim_{n\to\infty}\left(1 + \frac{k}{n}\right)^n = e^k,$$

showing that indeed, any positive number can be reached as a limit.
Example:

$$\lim_{n\to\infty}\frac{n^4 + n^6 + 28n^2}{219n^3 + n^{18}} = \lim_{n\to\infty}\frac{(n^4 + n^6 + 28n^2)/n^{18}}{(219n^3 + n^{18})/n^{18}} =$$

$$= \lim_{n\to\infty}\frac{n^{-14} + n^{-12} + 28n^{-16}}{219n^{-15} + 1} = \frac{0}{1} = 0,$$

because any negative power of n (as a sequence) tends to zero when n tends to infinity (same as positive powers of $\frac{1}{n}$). Here we apply the theorem about the sums and quotients of sequences with known limits.

1.4 Worked examples

1. What can we say about the following sequence, regarding monotonicity and boundedness?

$$a_n = \frac{2n}{3n+5}$$

Looking at the sequence, we can see that it has a limit: divide both the numerator and denominator with n, then we get $a_n = \frac{2}{3+\frac{5}{n}}$ that tends $\frac{2}{3}$ because $\frac{5}{n}$ tends to zero. So the sequence is convergent (tends to a finite real number). Using our theorems, it must be bounded. For finding an upper bound and a lower bound and also to check monotonicity, let's calculate a few values:

$$a_1 = \frac{2}{8} = 0.25 \quad a_2 = \frac{4}{11} \approx 0.364 \quad a_2 = \frac{6}{14} \approx 0.429 \text{ etc.}$$

This sequence can be monotonically increasing, that's what it looks like. Let's try to prove this property: $a_n < a_{n+1}$ means $\frac{2n}{3n+5} < \frac{2(n+1)}{3(n+1)+5}$, that is $\frac{2n}{3n+5} < \frac{2n+2}{3n+8}$. Or equivalently, $2n(3n+8) < (2n+2)(3n+5)$, if $3n+5 > 0$ and $3n+8 > 0$ that we need for the multiplication (if one of them is negative: the direction of the inequality will change; if one of them is zero: one of the original fractions makes no sense). The multipliers are both positive for any positive n, so what we wrote is correct. Let's expand the brackets:

$$6n^2 + 16n < 6n^2 + 6n + 10n + 10.$$

This is obviously true for any n, so we successfully proved that the sequence is monotonically increasing. Then finding a lower bound is easy, too: since the first term is 0.25, that yields a good lower bound (the greatest possible), and the limit ($\frac{2}{3}$) must be an upper bound (actually, the least upper bound) because of the monotonicity.

2. What can we say about the following sequence, regarding monotonicity and boundedness?

$$a_n = \frac{3n}{2n-5}$$

When we want to prove monotonicity here, multiplying by $2n-5$ will cause a problem until it becomes positive: from $n = 3$. So we can speak about monotonicity from a_3. Let's calculate a few elements. $a_3 = 9, \quad a_4 = 4, \quad a_5 = 3$, etc. So we can expect the sequence to be monotonically decreasing after $n = 3$. So $a_n > a_{n+1}$ for $n \geq 3$, this is what we try to prove. That means $\frac{3n}{2n-5} > \frac{3(n+1)}{2(n+1)-5}$, or equivalently, $3n(2n-3) > (3n+3)(2n-5)$, that is $6n^2 - 9n > 6n^2 + 6n - 15n - 15$ which is obviously true. So we have proven that the sequence is monotonically decreasing from $n = 3$.

Again, it is easy to see that the sequence is convergent: it tends to $\frac{3}{2}$. So it is surely bounded, according to an earlier theorem. Let's find a lower and an upper bound. We also have to calculate the first two elements:

$a_1 = \frac{3}{-3} = -1, \quad a_2 = \frac{6}{-1} = -6$. Knowing the limit and the monotonicity after $n = 3$ we can see that the least upper bound is 9 and the greatest lower bound is -6.

3. Prove the following statement based on the definition of convergence. Find the index N after which the distance of the elements of the sequence from the limit is smaller than the given ϵ.

$$\lim_{n\to\infty} \frac{2n^2-5}{6n^2-12n+3} = \frac{1}{3} \qquad \epsilon = 0.01$$

We have to find such an N that $|\frac{2n^2-5}{6n^2-12n+3} - \frac{1}{3}| < 0.01$ for all $n \geq N$.

We can see that $6n^2 - 12n + 3$ is smaller than $3(2n^2 - 5)$, but they are positive, even from $n = 2$. Increasing the denominator makes the fraction smaller, so $\frac{2n^2-5}{6n^2-12n+3}$ is greater than $\frac{1}{3}$. that means we can just leave the sign of the absolute value, and then we have the inequality $\frac{2n^2-5}{6n^2-12n+3} - \frac{1}{3} < 0.01$. Multiplying by 3 and also by $6n^2-12n+3$ yields $3(2n^2-5)-(6n^2-12n+3) < 0.03(6n^2-12n+3)$ if $6n^2-12n+3 > 0$. The latter requirement is true from $n = 2$, because $6n^2-12n+3 = n(6n-12)+3 = 6n(n-2)+3$ has a minimum at $n = 1$ and it is increasing for $n \geq 1$.

Let's rewrite what we have: $6n^2-15-6n^2+12n-3 < 0.18n^2-0.36n+0.09$, or equivalently: $12n-18 < 0.18n^2-0.36n+0.09$ or $0 < 0.18n^2-12.36n+18.09$.

Find its roots: $n_{1,2} = \frac{12.36\pm\sqrt{12.36^2-4\cdot 0.18\cdot 18.09}}{2\cdot 0.18}$. The inequality is fulfilled from the bigger root, which is approximately 67.2, so $N = 68$ is the smallest suitable N.

Let's check what we found: substitute $n = 70$ to $\frac{2n^2-5}{6n^2-12n+3}$: it is $\frac{2\cdot 4900-5}{6\cdot 4900-12\cdot 70+3} \approx 0.342926$, and its difference from $\frac{1}{3}$ is approximately 0.009593, which is indeed smaller than 0.01, as requested.

4. Find the limit of the following sequence.

$$a_n = \frac{3\cdot 4^n - 5\cdot 3^{n-1}}{2\cdot 2^{2n-1} + 4\cdot 3^{n+2}}$$

We have to evaluate the magnitude of each term. The multiplying constants are not important here (have no major effect), but the powers do: 4^n is greater than 3^n, and $2^{2n} = \left(2^2\right)^n = 4^n$. So the greatest term in both the numerator and denominator (in order of magnitude) is 4^n. Let's divide both the numerator and denominator by 4^n. We get the following: $\frac{3\cdot 1-\frac{5}{3}\cdot\left(\frac{3}{4}\right)^n}{\frac{2}{2}\cdot 1+4\cdot 9\cdot\left(\frac{3}{4}\right)^n}$. Now, since $\frac{3}{4} < 1$, its powers tend to zero. So the limit of the given expression is equal to $\frac{3}{1} = 3$.

5. Find the limit of the following sequence.

$$a_n = \left(\frac{n}{n+2}\right)^n$$

Since the limit of the expression in the bracket is 1, and its power tends to infinity, we are going to use the theorem about $\left(1+\frac{1}{n}\right)^n$ tending to e. Rephrasing the expression we get: $a_n = \left(1 - \frac{2}{n+2}\right)^{n+2} \cdot \left(1 - \frac{2}{n+2}\right)^{-2}$. Here the second factor tends to 1, because it is fixed finite power of an expression tending to 1. At the first factor, we can change the variable: let's denote $n+2=m$, then it becomes $\left(1-\frac{2}{m}\right)^m$, that tends to e^{-2}, as we mentioned. Let's prove it here, for this specific case. It's easier to start from the original form: $\left(\frac{n}{n+2}\right)^n$ is equal to the reciprocal of $\left(\frac{n+2}{n}\right)^n = \left(\left(\frac{\frac{n}{2}+1}{\frac{n}{2}}\right)^{\frac{n}{2}}\right)^2$. The expression inside the square tends to e, so its square tends to e^2, and its reciprocal tends to e^{-2}.

6. Find the limit of the following sequence.

$$a_n = \left(\frac{n+3}{3n}\right)^n$$

Here the expression in the bracket tends to zero, so its powers will also tend to zero. Formally the easiest proof is the following: $a_n = \left(\frac{n+3}{3n}\right)^n = \left(\frac{1}{3}\right)^n \cdot \left(\frac{n+3}{n}\right)^n$. Here the first factor tends to zero (the powers of a constant that is smaller than 1 in absolute value), and the second factor tends to e^3. So their product tends to zero as well.

1.5 Exercises

Consider the following sequences. After calculating a few of their elements, what can you say about them regarding monotonicity and boundedness?

$$a_n = \frac{n}{2n+1} \qquad a_n = \frac{1-2n}{2+2n}$$

$$a_n = 5n - 3 \qquad a_n = \left(-\frac{1}{5}\right)^n$$

$$a_n = \sin n\frac{\pi}{2} \qquad a_n = 7 - 2n$$

$$a_n = (-4)^n \qquad a_n = 3 \cdot 2^n$$

$$a_n = \frac{n^2+2n}{n+1} \qquad a_n = \frac{4n+1}{3^n}$$

Prove the following statements based on the definition of convergence. Find the index N after which the distance of the elements of the sequence from the limit is smaller than the given ϵ.

$$\lim_{n\to\infty} \frac{3n+2}{n+1} = 3 \qquad \epsilon = 0.02 \qquad\qquad \lim_{n\to\infty} \frac{2n-1}{3n+5} = \frac{2}{3} \qquad \epsilon = 0.1$$

$$\lim_{n\to\infty} \frac{6n-1}{2-3n} = -2 \qquad \epsilon = 10^{-2} \qquad\qquad \lim_{n\to\infty} \left(1 + \frac{(-1)^n}{n+2}\right) = 1 \qquad \epsilon = \frac{3}{100}$$

Find the limit of the following sequences, using the known limits (see them and the most important rules in section 1.3).

$$a_n = \frac{2n^2+5n-7}{n^2-5} \qquad\qquad a_n = \left(\frac{n+3}{n}\right)^n$$

$$a_n = \frac{5n^2-11n+8}{2n-7} \qquad\qquad a_n = \frac{7n-3}{2n^2+1}$$

$$a_n = \frac{3n^2-5}{-n^2+2n}\cdot\left(2 - \frac{5}{n-2} + \frac{4n-1}{-n+3}\right) \qquad\qquad a_n = \frac{-8n+6}{n-13}$$

$$a_n = \frac{2n-1}{3n+5}\cdot\left(4 - \left(-\frac{1}{2}\right)^n + \frac{5n^2-2n}{7n^3+5n-1}\right) \qquad\qquad a_n = \frac{3\cdot 5^{n-1}+2\cdot 3^n}{4^{n+2}+5\cdot 3^{n-2}}$$

$$a_n = \frac{3^{n-1}}{2^n+4^{n+1}} \qquad\qquad a_n = \frac{4\cdot 2^{n-1}+3\cdot(-7)^n}{2\cdot 3^{n+1}+5\cdot 4^n}$$

$$a_n = \frac{(-3)^n+3\cdot 2^{n-1}}{3^{n-1}+9} \qquad\qquad a_n = \frac{7\cdot 9^n-6\cdot 5^{n-1}}{5\cdot 4^n-2\cdot 3^{2n}}$$

$$a_n = \frac{2\cdot 3^n-5\cdot 4^{n+1}}{4\cdot 2^{n-1}+5\cdot 3^{n+2}} \qquad\qquad a_n = (-1)^n\cdot\frac{2n+3}{n^2-7}$$

$$a_n = 1 - \frac{(-1)^n}{n} \qquad\qquad a_n = (-1)^n\cdot\frac{3n^2+n}{2n-1}$$

$$a_n = (-1)^n\cdot\frac{7n-5}{n+9} \qquad\qquad a_n = \left(\frac{n-1}{2n}\right)^n$$

Chapter 2

Basic functions

2.1 The image of basic functions

When we speak about a *function* in this chapter, we mean a function mapping real numbers to real numbers. Let $\mathcal{D} \subset \mathbb{R}$ be a set of real numbers, and for each element $x \in \mathcal{D}$ define $f(x)$, the image of x uniquely. $\mathcal{D}(f)$ will be called the *domain* of the function f, and the set $\mathcal{R}(f)$ of those values that are taken as images of numbers in $\mathcal{D}$ will be called the *range* of f.

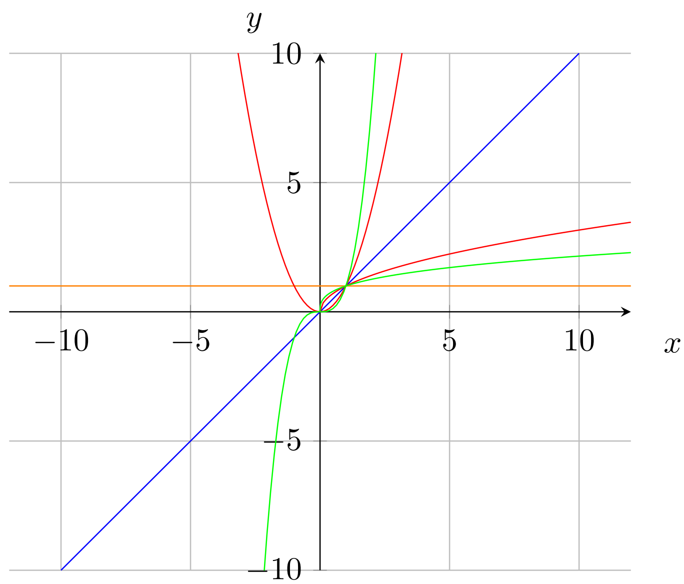
The *inverse* of a function reverses the direction of the mapping: it maps the elements of $\mathcal{R}(f)$ to the elements of $\mathcal{D}(f)$. A function f only has an inverse if the mapping from $\mathcal{D}(f)$ to $\mathcal{R}(f)$ in one-to-one: each point in $\mathcal{R}(f)$ only corresponds to one point in $\mathcal{D}(f)$.

2.1.1 Powers of x

The most important functions are the *powers* of x: x^k, $k \in \mathbb{R}$. *Polynomials* are linear combinations of non-negative integer powers of x, with real coefficients, for example $6.43x^7 + 5x^3 - 2.1x^2 + 14$. *Rational functions* are those that can be defined by a fraction such that both the numerator and the denominator are polynomials, e.g. $\frac{5x^2+6.2}{7.3x^3+4x^2+7.01}$.

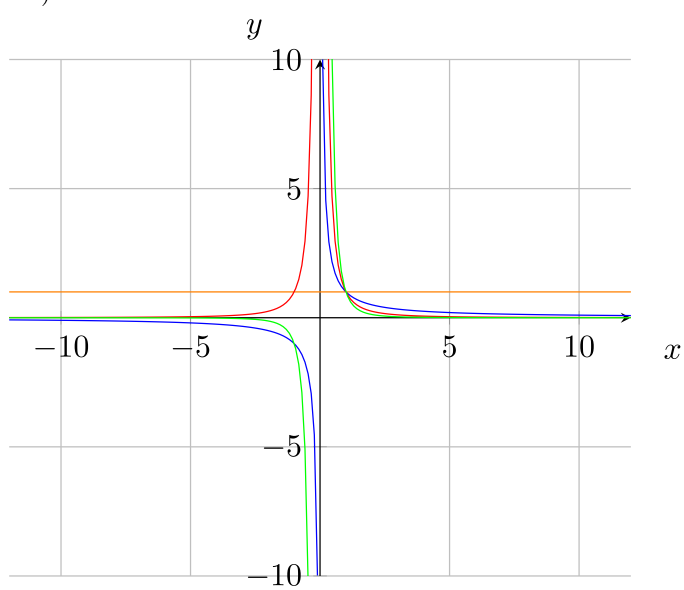
We can plot the pairs of values $(x, f(x))$ in a coordinate system (we can write $y = f(x)$). This plot will be called the *graph* of the function f. You must remember what the graphs of the following functions look like.

Positive (and $0th$) powers of x are depicted in the first image:
$y = x^0 = 1$ (orange), $y = x$ (blue), $y = x^2$ (red), $y = x^3$ (green),
$y = \sqrt{x}$ (red), $y = \sqrt[3]{x}$ (green).



Negative (and $0th$) powers of x are depicted in the second image:

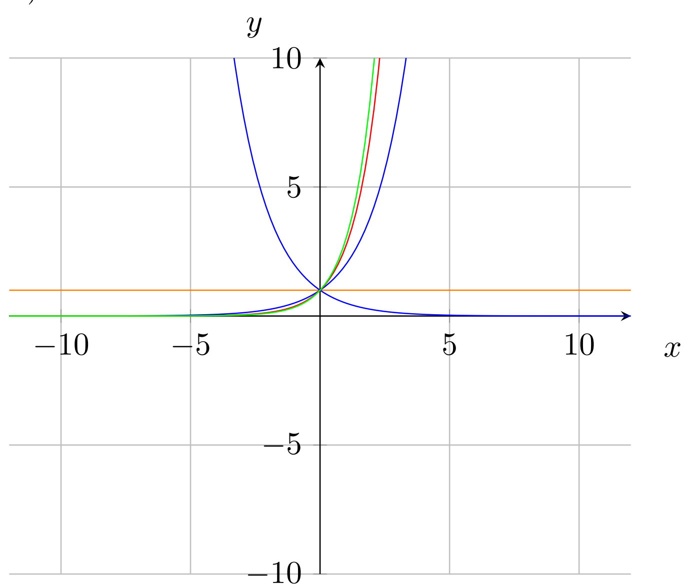
$y = x^0 = 1$ (orange), $y = x^{-1} = \frac{1}{x}$ (blue), $y = x^{-2} = \frac{1}{x^2}$ (red), $y = x^{-3} = \frac{1}{x^3}$ (green).



2.1.2 Exponential functions

Exponential functions are depicted in the third image:

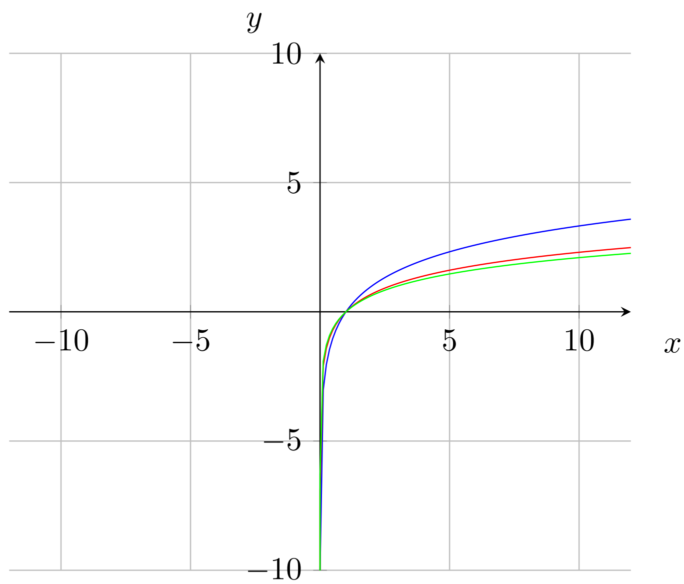
$y = 1^x = 1$ (orange), $y = 2^x$ (blue), $y = e^x$ (red), $y = 3^x$ (green), $y = \left(\frac{1}{2}\right)^x$ (blue).



2.1.3 Logarithmic functions

Logarithmic functions are the inverse of exponential functions. (For example, we say that the *inverse* of the function 2^x is $\log_2 x$, because if $y = 2^x$, then $x = \log_2 y$.) Some examples for logarithmic functions are depicted in the 4th image:

$y = \log_2 x$ (blue), $y = \log_e x$ (red), $y = \log_3 x$ (green).

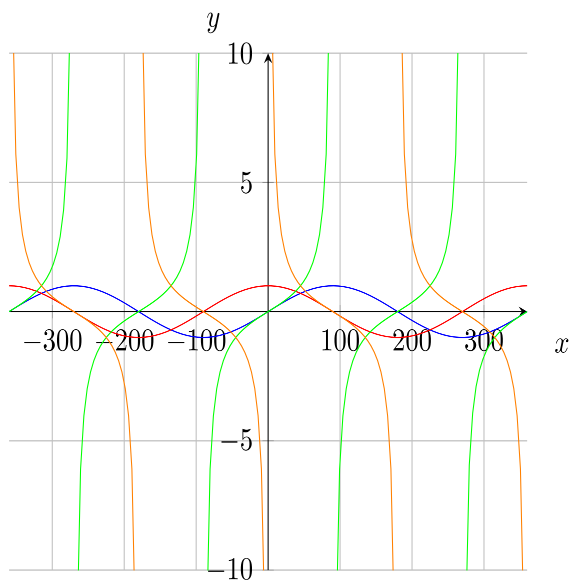


2.1.4 Trigonometric functions

Originally, trigonometric functions are defined on a right-angled triangle, where the other two angles are sharp angles. If the hypotenuse is 1, the length of the side opposite to α is $\sin\alpha$ and the length of the adjacent side is $\cos\alpha$. Now put the triangle in a coordinate system so that the vertex at the angle α shall be the Origin, and the direction of the adjacent side shall be the positive direction of the x axis. Keep the hypotenuse equal to 1, but open the angle α so that the other endpoint of the hypotenuse (P) goes along a circle of radius 1. Now $\sin\alpha$ will be the 2nd coordinate of the point P, and $\cos\alpha$ will be the 1st coordinate of the point P. It is obvious from this construction that $\cos\alpha$ is negative if α is between 90° and 270°, and $\sin\alpha$ is negative if α is between 180° and 360°.

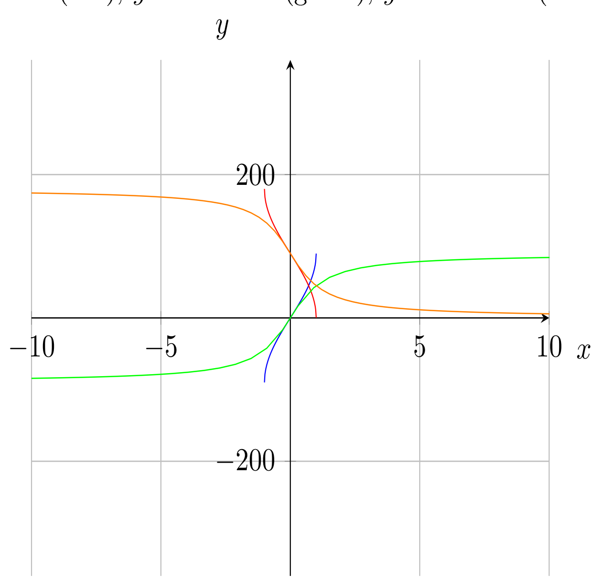
Tangent of α is defined as $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$, and its cotangent is defined as $\cot\alpha = \frac{\cos\alpha}{\sin\alpha}$.

Their graph is plotted in the following image. $y = \sin x$ (blue), $y = \cos x$ (red), $y = \tan x$ (green), $y = \cot x$ (orange).



The inverse of $\sin\alpha$ is called $\arcsin x$, the inverse of $\cos\alpha$ is called $\arccos x$, the inverse of $\tan\alpha$ is called $\arctan x$ and the inverse of $\cot\alpha$ is called arccot x. They can only be defined where the original function is monotonic (either monotonically increasing, or monotonically decreasing), so arcsin x and arccos x are defined only in a part of the real line.

Their graph is plotted in the following image. $y = \arcsin x$ (blue), $y = \arccos x$ (red), $y = \arctan x$ (green), $y =$ arccot x (orange).



2.2 Limit of functions

The *limit of a function* $f(x)$ at $x = x_0$ can be defined similarly as the limit of a *sequence* was defined. Take a sequence of numbers x_n tending to x_0, within the domain of the function f. Now if the sequence $f(x_n)$ converges to $A \in \mathbb{R}$, and this is true for *any* sequence of numbers (x_n) tending to x_0, then we say that $\lim_{x \to x_0} f(x) = A$.

We say that a function f is *continuous* if $\lim_{x \to x_0} f(x) = f(x_0)$ for any x_0 in its domain.

2.3 Convexity

Definition: we say that a function $f(x)$ is *convex* in a given interval if it is continuous there, and for any x_1 and x_2 in that interval

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}.$$

The geometric meaning of a function being convex is, that if we connect two points of its graphs with a straight line, the connecting line passes above the graph of the function. You will see an example for the meaning of convexity in the next section.

If the opposite is true (the connecting line passes below the graph of the function), we say the function is *concave*:

$$f\left(\frac{x_1 + x_2}{2}\right) \geq \frac{f(x_1) + f(x_2)}{2}.$$

2.4 Plotting a function

Knowing how to calculate limits, we are already able to see what a function looks like approximately. It is also useful to calculate where the function crosses the x and y axis.

Example: Sketch the graph of the function $f(x) = \frac{x-1}{5(x+1)(x+2)}$.

Solution: The domain of the function is $\mathbb{R} \setminus \{-2\} \setminus \{-1\}$.

Calculating the limits at the borders of the domain:

a) $\lim_{x \to -\infty} = 0$ (when x tends to $-\infty$): same technique as we used at sequences, when n tends to ∞, just you have to be careful with the signs.

b) $\lim_{x \uparrow -2} = -\infty$ (when x tends to -2 from below): it obviously tends to infinity because the denominator is tending to zero, but you have to decide its sign, that can make it $+\infty$ or $-\infty$.

c) $\lim_{x \downarrow -2} = +\infty$ (when x tends to -2 from above): it obviously tends to infinity because the denominator is tending to zero, but you have to decide its sign, that can make it $+\infty$ or $-\infty$.

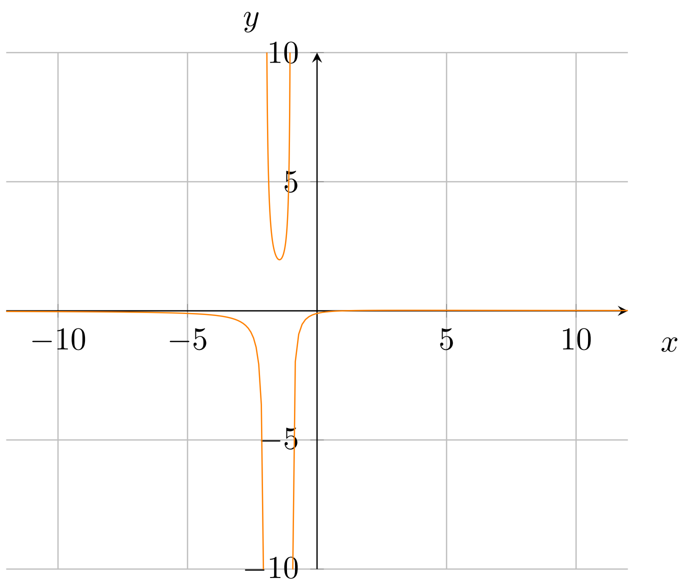
d) $\lim_{x\uparrow -1} = +\infty$ (when x tends to in -1 from the left): it obviously tends to infinity because the denominator is tending to zero, but you have to decide its sign, that can make it $+\infty$ or $-\infty$.

e) $\lim_{x\downarrow -1} = -\infty$ (when x tends to in -1 from the right): it obviously tends to infinity because the denominator is tending to zero, but you have to decide its sign, that can make it $+\infty$ or $-\infty$.

h) $\lim_{x\to +\infty} = 0$ (when x tends to $+\infty$): same technique as we used at sequences, when n tends to ∞.

Substituting $x = 0$: $f(0) = \frac{-1}{5(+1)(+2)} = \frac{-1}{10}$. This is where the function crosses the y axis.

Finding the roots of $f(x)$: where $f(x) = 0$, that is where the function crosses the x axis. This is at $x = 1$.



Apparently, the function is concave in $(-\infty, -2)$, convex in $(-2, -1)$, and concave in $(-1, \infty)$.

2.5 Exercises

1. Sketch the graph of the function $f(x) = \sin \frac{1}{x}$.
2. Sketch the graph of the function $f(x) = \cos \frac{1}{x}$.
3. Sketch the graph of the function $f(x) = \cosh x = \frac{e^x + e^{-x}}{2}$.
4. Sketch the graph of the function $f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$.

(You can check your solutions using the website *www.wolframalpha.com*.)

Chapter 3

Differentiation

Differentiation is a tool to get a more refined picture about curves (graphs of functions).

3.1 Definition

1) *Graphic representation:*

The derivative of a function $f(x)$ is an other function $f'(x)$, whose value in $x = x_0$ is equal to the *slope of the tangent line* to the graph of the function $f(x)$ at the point $(x_0, f(x_0))$. Obviously, the tangent line is the extremum of secant lines going through the point $(x_0, f(x_0))$. This will enable us to use limits in the following algebraic definition.

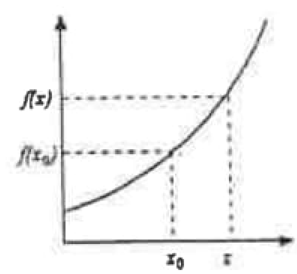
2) *Formal definition:*

Let $f(x)$ be defined in a neighbourhood of x_0.

a) We are going to use the following *differences:* $x - x_0$ and $f(x) - f(x_0)$.

b) Difference quotients of the function $f(x)$ at x_0 are of the form

$$\frac{f(x) - f(x_0)}{x - x_0},$$



where $x \neq x_0$. For a fixed x_0, they can be regarded as a function of x.

c) If the difference quotient, as a function of x, has a limit at x_0, then we say $f(x)$ is differentiable at x_0, and the limit of the difference quotient is the derivative of $f(x)$ at x_0:

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} =: f'(x_0).$$

$f'(x_0)$ is the *derivative* of the function $f(x)$ at $x = x_0$.

Now let $f(x)$ be defined on an interval (a, b).

a) Let's use the following notation for the differences in the variable x and the corresponding values of the function f:

Δx and $f(x + \Delta x) - f(x)$.

b) With this notation, a difference quotient of the function $f(x)$ at x is of the form

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

c) If the difference quotient has a limit for any $x \in (a, b)$ as Δx is tending to zero, then we say $f(x)$ is differentiable in the interval (a, b), and for any $x \in (a, b)$:

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x).$$

Then $f'(x)$ is a function defined in the interval (a, b), which is called the derivative of the function $f(x)$.

3) Examples for quotient of functions both tending to zero, where the quotient has a limit:

$$\lim_{x\to 2}\frac{x^2-2^2}{x-2} = \lim_{x\to 2}\frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2}(x+2) = 2+2 = 4,$$

$$\lim_{x\to x_0}\frac{x^2-x_0^2}{x-x_0} = \lim_{x\to x_0}\frac{(x+x_0)(x-x_0)}{x-x_0} = \lim_{x\to x_0}(x+x_0) = x_0+x_0 = 2x_0,$$

(The latter argument is a proof for the derivative of $f(x) = x^2$ being $2x$.)

3.2 Derivative of elementary functions

a) Derivative of a constant is zero:

$$c' = 0.$$

b) Derivative of power functions:

$$(x^r)' = r \cdot x^{r-1} \qquad (r \in \mathbb{R})$$

c) Derivative of exponential functions:

$$(e^x)' = e^x$$

$$(a^x)' = a^x \ln a$$

d) Derivative of logarithmic functions:

$$(\ln x)' = \frac{1}{x}$$

$$(\log_a x)' = \frac{1}{x \ln a}$$

e) Derivative of trigonometric functions:

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

$$(\tan x)' = \frac{1}{(\cos x)^2} = \frac{1}{\cos^2 x} = 1 + \tan^2 x$$

$$(\cot x)' = \frac{-1}{(\sin x)^2} = \frac{-1}{\sin^2 x} = -(1 + \cot^2 x)$$

f) Derivative of the inverse of trigonometric functions:

$$(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}$$

$$(\arccos x)' = \frac{-1}{\sqrt{1 - x^2}}$$

$$(\arctan x)' = \frac{1}{1 + x^2}$$

$$(\operatorname{arccot} x)' = \frac{-1}{1 + x^2}$$

3.3 The inverse of a function

Definition: $f(x) = y$ if and only if $f^{-1}(y) = x$, where f^{-1} is a notation for the inverse of f. We can say that the inverse of f "reverses" the action of f.

Strictly monotonic functions always have an inverse. Some functions do not have an inverse over their whole domain, but there exists a part of their domain where they are invertible. E.g. $f(x) = x^2$ has an inverse for half of its domain: restrict the function $f(x) = x^2$ to non-negative numbers, there it is strictly monotonic, and it has an inverse: $y = x^2$ is equivalent to $x = \sqrt{y}$ here.

Geometric meaning: the graph of the inverse of $f(x)$ is the same as the graph of $f(x)$, we just have to swap the x and y axis!

3.4 Rules of differentiation

If the functions f and g are differentiable at x, then

1. their constant multiples are also differentiable, and $(cf(x))' = cf'(x)$, where $c \in \mathbb{R}$;
2. their sum $f + g$ is also differentiable, and $(f(x) + g(x))' = f'(x) + g'(x)$;
3. their product $f \cdot g$ is also differentiable, and $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$;
4. if $g(x) \neq 0$, then their quotient $\frac{f}{g}$ is also differentiable, and

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

If the function g is differentiable at x, and the function f is differentiable at $g(x)$, then their composition $f \circ g$ is also differentiable at x, and

$$(f \circ g)'(x) = (f(g(x)))' = f'(g(x)) \cdot g'(x).$$

How to prove the above statements? See the poofs below, based on the definition of the derivative.

1. Constant multiplier can be factored out:

$$(cf(x))' = cf'(x)$$

Proof:

$$\lim_{x \to x_0} \frac{cf(x) - cf(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{c(f(x) - f(x_0))}{x - x_0} = c \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = cf'(x)$$

2. Derivative of the sum of functions is the sum of the derivatives:

$$(f(x) + g(x))' = f'(x) + g'(x)$$

Proof:

$$\lim_{x \to x_0} \frac{(f(x) + g(x)) - (f(x_0) + g(x_0))}{x - x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0} = f'(x_0) + g'(x_0)$$

3. Derivative of product of functions (product rule):

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

Proof:

$$\lim_{x\to x_0}\frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0}=\lim_{x\to x_0}\frac{f(x)g(x)-f(x)g(x_0)+f(x)g(x_0)-f(x_0)g(x_0)}{x-x_0}=$$

$$=\lim_{x\to x_0} f(x)\frac{g(x)-g(x_0)}{x-x_0}+g(x_0)\frac{f(x)-f(x_0)}{x-x_0}=f(x_0)g'(x_0)+g(x_0)f'(x_0).$$

Example:

$$(x^4\sin x)'=4x^3\sin x+x^4\cos x$$

4. Quotient rule:

$$\left(\frac{f(x)}{g(x)}\right)'=\frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$$

It will be proven using the product rule and the chain rule (see the next point), for the product $f(x)(g(x))^{-1}$:

$$\left(f(x)(g(x))^{-1}\right)'=f'(x)(g(x))^{-1}+f(x)\left((g(x))^{-1}\right)'=f'(x)(g(x))^{-1}+f(x)(-1)(g(x))^{-2}g'(x)=$$

$$=\frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}.$$

Example:

$$\left(\frac{\sin x}{\cos x}\right)'=\frac{\cos x\cos x-\sin x(-\sin x)}{\cos^2 x}=\frac{1}{\cos^2 x}$$

5. Derivative of the composition of functions (chain rule):

$$f(g(x))'=f'(g(x))g'(x)$$

Proof:

$$\lim_{x\to x_0}\frac{f(g(x))-f(g(x_0))}{x-x_0}=\lim_{x\to x_0}\frac{f(g(x))-f(g(x_0))}{g(x)-g(x_0)}\cdot\frac{g(x)-g(x_0)}{x-x_0}=$$

$$=\lim_{g(x)\to g(x_0)}\frac{f(g(x))-f(g(x_0))}{g(x)-g(x_0)}\cdot\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0}=f'(g(x_0))\cdot g'(x_0).$$

Example:

$$(\sin(x^4))'=\cos(x^4)\cdot 4x^3.$$

3.5 Applications

Finding minimum / maximum
The derivative of a function tells about its tendency: direction of change, and speed of change. If the function is increasing, the derivative must be positive or zero. If the function is decreasing, the derivative must be negative or zero.
Where the function has a local minimum, it is decreasing till the minimum point, then increasing after that. So the derivative is non-positive before the minimum point, and non-negative after that (in a neighbourhood at least).
That means, if the derivative is continuous, it must be zero at the minimum point.
Similarly, where the function has a local maximum, it changes from increasing to decreasing. Its derivative is changing from non-negative to non-positive (in a neighbourhood at least). So again, if the derivative is continuous, it must be zero at the maximum point.
We can see that the place where the function has zero derivative, is a possible local minimum or maximum.

Equation of the tangent line
If the function $f(x)$ is differentiable at x_0 and it is continuous in a neighbourhood of x_0, then the *equation of the tangent line* to the graph of the function is the straight line $y - f(x_0) = f'(x_0)(x - x_0)$, using that $f'(x_0)$ is the slope of the line tangent to the graph of the function at the point $(x_0, f(x_0))$.

3.6 Worked examples

Basic derivatives collected in a table:

Function	Its derivative	Restrictions
$y = const.$	$y' = 0$	-
$y = x^n$	$y' = nx^{n-1}$	$n \in \mathbb{N}, x \in \mathbb{R}$
$y = x^r$	$y' = rx^{r-1}$	$r \in \mathbb{R}, x > 0$
$y = a^x$	$y' = a^x \ln a$	$x \in \mathbb{R}, a \in \mathbb{R}, a > 0$
spec. $y = e^x$	$y' = e^x$	$x \in \mathbb{R}$
$y = \log_a x$	$y' = \frac{1}{x \ln a}$	$x > 0, a \in \mathbb{R}, a > 0, a \neq 1$
spec. $y = \ln x$	$y' = \frac{1}{x}$	$x > 0$
$y = \sin x$	$y' = \cos x$	$x \in \mathbb{R}$
$y = \cos x$	$y' = -\sin x$	$x \in \mathbb{R}$
$y = \tan x$	$y' = \frac{1}{\cos^2 x}$	$x \neq \frac{\pi}{2} + k\pi$
$y = \cot x$	$y' = -\frac{1}{\sin^2 x}$	$x \neq k\pi$
$y = \arcsin x$	$y' = \frac{1}{\sqrt{1-x^2}}$	$\lvert x\rvert < 1$
$y = \arccos x$	$y' = -\frac{1}{\sqrt{1-x^2}}$	$\lvert x\rvert < 1$
$y = \arctan x$	$y' = \frac{1}{1+x^2}$	$x \in \mathbb{R}$
$y = \text{arccot}\, x$	$y' = -\frac{1}{1+x^2}$	$x \in \mathbb{R}$

Example 1. Application of $(f \pm g)' = f' \pm g'$; $(fg)' = f'g + fg'$; $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$:

Function $(f(x))$	Its derivative $(f'(x))$
$4x^3 + \sqrt[3]{x}$	$(4x^3)' + (x^{\frac{1}{3}})' = 4 \cdot 3x^2 + \frac{1}{3}x^{-\frac{2}{3}}$
$(5x+2)\ln x$	$(5x+2)' \ln x + (5x+2)(\ln x)' = 5 \ln x + (5x+2)\frac{1}{x}$
$\frac{x^3+x^2+x}{\cos x}$	$\frac{(x^3+x^2+x)' \cos x - (x^3+x^2+x)(\cos x)'}{(\cos x)^2} = \frac{(3x^2+2x+1)\cos x - (x^3+x^2+x)(-\sin x)}{\cos^2 x}$

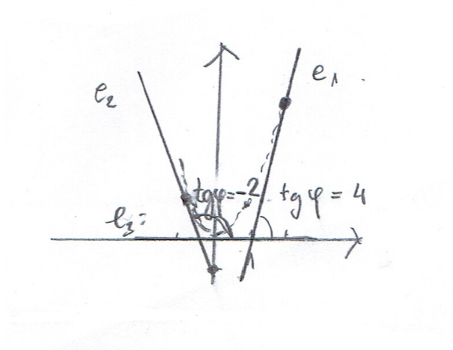
Example 2.

Composite function	Its derivative
$f(x) = (2x^5 + \sqrt{x})^3$	$f'(x) = 3(2x^5 + \sqrt{x})^2 \cdot (10x^4 + \frac{1}{2}x^{-\frac{1}{2}})$
$f(x) = \tan x^3 = \tan(x^3)$	$f'(x) = \frac{1}{\cos^2(x^3)} 3x^2$
$f(x) = \tan^3 x = (\tan x)^3$	$f'(x) = 3(\tan x)^2 \frac{1}{\cos^2 x}$

Example 3.
Find the equation of the line tangent to the graph of the function $f(x) = x^2$ at the points belonging to the values $x_0 = 2$, $x_0 = -1$ and $x_0 = 0$.

The equation of the tangent line is given by the following formula:
$y - f(x_0) = f'(x_0)(x - x_0)$, where now we have $f(x) = x^2$ and $f'(x) = 2x$.

Value of x_0	The equation of the tangent line (calculation)	Simpler form
2	$y - 2^2 = 2x\big\|_{x_0=2}(x-2) = 2 \cdot 2(x-2) = 4x - 8$	$e_1 : \; y = 4x - 4$
-1	$y - (-1)^2 = 2x\big\|_{x_0=-1}(x-(-1)) = -2(x+1) = -2x - 2$	$e_2 : \; y = -2x - 1$
0	$y - 0^2 = 2x\big\|_{x_0=0}(x-0)) = 0 \cdot (x-0) = 0$	$e_3 : \; y = 0$



3.7 Exercises

Find the derivative of the following functions (1 - 30.):

1. $f(x) = 4x^3 - x^2 + 7$
2. $f(x) = x^4 - 2x^2 + 7x + 6$
3. $f(x) = 4x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6$
4. $f(x) = 4x^{\frac{3}{2}} - 3\sqrt{x}$
5. $f(x) = 4x^{\frac{3}{2}} - 3\sqrt{9x}$
6. $f(x) = \frac{3}{x} - 5x^{\frac{5}{3}} + 7\sqrt[3]{x}$
7. $f(x) = (2x+5)(3x^7 - 8x^2)$
8. $f(x) = (5x+7)\sqrt{4x^5}$
9. $f(x) = (3x^7 - 8x^2)\sin x$
10. $f(x) = (3x^3 - 8x^2)(\sin x - \cos x)$
11. $f(x) = \frac{x^3-1}{1+2x}$
12. $f(x) = \frac{x^3+4}{x^2+2x}$
13. $f(x) = \frac{x^3+x^2+x}{\cos x}$
14. $f(x) = \frac{x^2 \tan x}{2+\cos x}$
15. $f(x) = \frac{4}{(1-x^2)(1-3x^3)}$
16. $f(x) = \frac{x^3+3}{(x^2+2x+3)(\sin x)}$
17. $f(x) = \frac{2x^2-4}{(1-x^2)\sqrt{x}}$
18. $f(x) = \frac{1-\arcsin x}{1+\arcsin x}$
19. $f(x) = x^3 \ln x$
20. $f(x) = 3^x(3x^6 - 8x^2 + 2)$

21. $f(x) = e^x(3x^2 - 4x)$
22. $f(x) = x \cdot \sin x \cdot \ln x$
23. $f(x) = 2^x \cdot \sin x \cdot \log_3 x$
24. $f(x) = \sin^3 x$
25. $f(x) = \sin x^3$
26. $f(x) = \tan(4x^2 + 1)$
27. $f(x) = \sin(x^2 + 3x + 4)$
28. $f(x) = \sqrt[3]{x - 3x^5}$
29. $f(x) = \frac{1}{\cos 5x}$
30. $f(x) = (3x^7 - 8x^2)^{10}$

31. Find the equations of the tangent lines to the graph of the function $y = \sin x$ at the points defined by $x_0 = 0,\ \frac{\pi}{2},\ \pi$ and 2π. Also prepare a (correct) image.

Chapter 4

Integration

In our practice, we have plenty of tasks, problems when we need integration for the exact answer: calculating *areas, volumes* or the *centre of mass.*

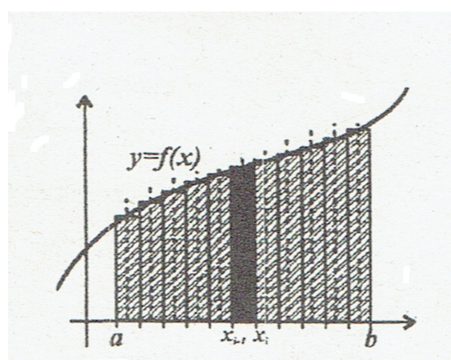
Later, when we have solved such problems using integration, let us pose the question to ourselves: How could I calculate the answer without knowing integration? How much should I work with it then? Would there be any solution? Could I give an exact result? – If you can answer these questions, then you surely know how the notion of integrals appeared, – what is integration.

Definite integrals.
Let $f(x)$ be defined everywhere on the interval $[a, b]$.
By the *definite integral* of $f(x)$ from a to b we mean the result of the following approach: the limit gained through the following approximation.

- Divide the interval $[a, b]$, proceeding from a to b, with the values $x_1, x_2, ..., x_{n-1}$, and let $a = x_0$, $b = x_n$. *Note:* $[a, b]$ can be divided to equal or non-equal parts.

- Let us select an element (the variable ξ_i) from each segment gained: $\xi_i \in [x_{i-1}, x_i]$, $i = 1, 2, ..., n$.

- Consider the value of the function at each ξ_i taken from the segment, and form the product $f(\xi_i)(x_i - x_{i-1})$, or with shorter notation $f(\xi_i)(x_i - x_{i-1}) = f(\xi_i)\Delta x_i$.



- Summarising the values of the above products:
  $S_n = f(\xi_1)\Delta x_1 + f(\xi_2)\Delta x_2 + ... + f(\xi_n)\Delta x_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i$.
- If the sequence of *integral-approximating sums* S_n has a limit − when $[a, b]$ is divided into more and more parts ($n \to \infty$) and the division is more and more refined ($\max_i |\Delta x_i| \to 0$), − and this limit is independent of the segmentation of $[a, b]$ and the choice of the values ξ_i, then we call the limit the *definite integral of* $f(x)$ from a to b and denote it by the symbol $\int_a^b f(x)dx$, so

$$\lim_{\substack{n \to \infty \\ \max_i |\Delta x_i| \to 0}} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \int_a^b f(x)dx.$$

Note:

- It may be a bit confusing that a is called the "lower limit of the integral" and b is the "upper limit of the integral" in $\int_a^b f(x)dx$.
- If you encounter an integral where the lower limit is greater than the upper limit, it can be defined as $\int_b^a f(x)dx = -\int_a^b f(x)dx$, that is equivalent to understanding the above definition with the division proceeding from (the greater) b to (the smaller) a, so that the values $x_i - x_{i-1}$ $(i = 1, 2, ..., n)$ are negative.

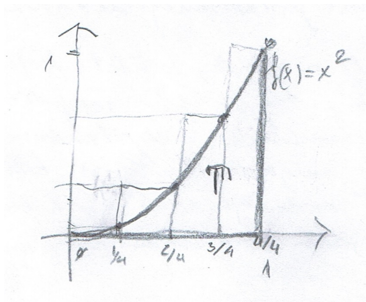
Primitive function, indefinite integral.

a) Primitive function.
Let $f(x)$ and $F(x)$ be defined on an interval and $F(x)$ be differentiable. If $F'(x) = f(x)$, then $F(x)$ is called a *primitive function* of $f(x)$.
If $f(x)$ has a primitive function, then it has infinitely many primitive functions that differ in an additive constant only. For we have
$(F_1(x) - F_2(x))' = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$, so $F_1(x) - F_2(x) = c$.
Note: If the domain of the function is not connected, then this constant can be different on the components of the domain.

b) Indefinite integral.
All the primitive functions of $f(x)$ together are called the *indefinite integral* of $f(x)$ and denoted by $\int f(x)dx$.

The two different notions of integral are connected by the *Newton-Leibniz theorem*, whose exact form you will find after the following example that shows the connection.

Example 1.
Consider the area T that you can see in the following image, enclosed by the function $f(x) = x^2$ and the x axis. Let T_1 denote an approximate calculation for T, and T_2 the exact calculation of the area T.



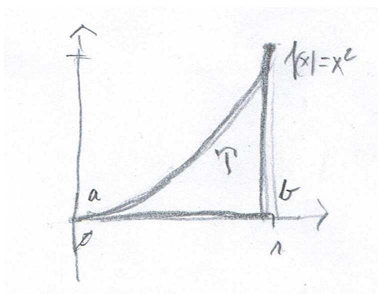
$T_1 = f(\frac{1}{4})(\frac{1}{4} - 0) + f(\frac{2}{4})(\frac{2}{4} - \frac{1}{4}) + ... +$
$f(\frac{4}{4})(\frac{4}{4} - \frac{3}{4}) =$
$= (\frac{1}{4})^2 \cdot \frac{1}{4} + (\frac{2}{4})^2 \cdot \frac{1}{4} + (\frac{3}{4})^2 \cdot \frac{1}{4} + (\frac{4}{4})^2 \cdot \frac{1}{4}$

Question: Can T be calculated?

Answer: Yes, with different approximations.

Question: $T_1 = T$?

Answer: No, $T_1 \approx T$.



$T_2 = \int_a^b f(x)dx = \int_0^1 x^2 dx$
Question: How can we calculate it?
Answer: using the Newton-Leibniz theorem.
Question: $T = \int_0^1 f(x)dx = \int_0^1 x^2 dx$?
Answer: Yes, as a conclusion of the definition of the definite integral.

We already know that the definite integral determines an area as the limit of approximating sums, but we don't know yet how can we find this limit (if it exists). We have to know the Newton-Leibniz theorem for the solution.

Newton-Leibniz theorem: If $f(x)$ and $F(x)$ are functions of one variable, continuous on the interval $[a, b]$, and $F(x)$ is a primitive function of $f(x)$, that is, $F(x)$ is differentiable on (a, b) and $F'(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a) = [F(x)]_a^b,$$

that is *the change of $F(x)$ from a to b.*

Note: We don't prove the Newton-Leibniz theorem here, but you can convince yourself of its truth easily in a geometric way: consider an integral-approximating sum from $-\infty$ to x, and its limit, the integral on a half-line. This is one of the primitive functions of $f(x)$; at differentiation we cut off a "slice" from the area, divide the area of the slice by $\Delta_i = x_i - x_{i-1}$, so we get the hight of the slice, that equals the value of the function in a point between x_i and x_{i-1}, so it is approximately $f(x_i)$.

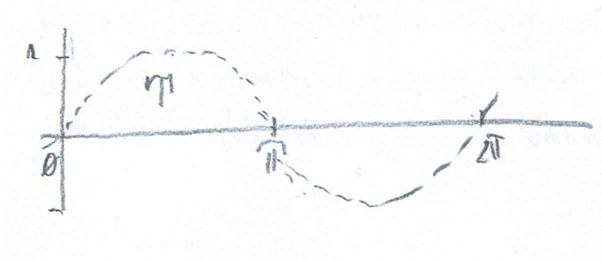
Let us return to the calculation of $\int_0^1 x^2 dx$:
$f(x) = x^2$
$F(x) = \frac{1}{3}x^3$ is suitable because $F'(x) = (\frac{1}{3}x^3)' = \frac{1}{3}3x^2 = x^2$.
Note: x^2 has infinitely many primitive functions because
$(\frac{1}{3}x^3 + c)' = (\frac{1}{3}x^3)' + (c)' = x^2 + 0$ if $c \in \mathbb{R}$.

$\int_a^b f(x) = F(b) - F(a)$ according to the Newton-Leibniz theorem, so

$$\int_0^1 x^2 dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}x^3\big|_{x=1} - \frac{1}{3}x^3\big|_{x=0} = \frac{1}{3}1^3 - \frac{1}{3}0^3 = \frac{1}{3}.$$

So the area under the curve of the function $f(x) = x^2$ over the interval $[0, 1]$ is $T = \frac{1}{3}$.

Example - 2.
Consider the function $f(x) = \sin x$ and determine the area under its curve over the interval $[0, \pi]$, denoted by T.



Solution:
$T = \int_0^\pi \sin x dx$
The function is $f(x) = \sin x$.
The primitive function of $f(x)$ is $F(x) = -\cos x$ because
$F'(x) = (-\cos x)' = \sin x$.
Applying the theorem $\int_a^b f(x)dx = F(b) - F(a)$,
$\int_0^\pi \sin x\, dx = [-cosx]_0^\pi = (-cos\pi) - (-cos0) = 1 - (-1) = 2.$
So the area under the curve of the function in question is $T = 2$.

Example - 3.
Calculate the following integrals.

a) $\int_\pi^{2\pi} \sin x\, dx$

b) $\int_0^{2\pi} \sin x\, dx$

Solution:

a) $\int_\pi^{2\pi} \sin x\, dx = \big[-\cos x\big]_\pi^{2\pi} = (-\cos 2\pi) - (-\cos\pi) = -1 - (+1) = -2$

b) $\int_0^{2\pi} \sin x\, dx = \big[-\cos x\big]_0^{2\pi} = (-\cos 2\pi) - (-\cos 0) = -1 - (-1) = 0$

Explanation of the results:

a) The function $f(x) = \sin x$ is negative everywhere on the interval $[\pi, 2\pi]$.

b) The function $f(x) = \sin x$ encloses the same size of area over the interval $[0, 2\pi]$ below and above the x axis, but the integrals have opposite signs because of the behaviour of the function $f(x) = \sin x$. ($f(x) \geq 0$ if $x \in [0, \pi]$, and $f(x) \leq 0$ if $x \in [\pi, 2\pi]$.)

Theorem: If $f(x) \geq 0$ is an integrable function defined on the interval $x \in [a, b]$, then the area under the curve of the graph of the function is

$$T = \int_a^b f(x)dx.$$

The consequence of the theorem is that if $f(x) \geq 0$ is not true over the whole interval $x \in [a, b]$, then we can calculate the *signed area* by the integral $\int_a^b f(x)dx$ (signed areas summed).

The definite integral has many different applications (we will see a few examples), but we need some experience first in finding the indefinite integral (a primitive function).

Examples:

- $\int(6x^2 - \sqrt{x})dx = 6\frac{x^3}{3} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = 2x^3 - \frac{2}{3}x^{\frac{3}{2}} + c$, where $c \in \mathbb{R}$.

  Check: $\left(2x^3 - \frac{2}{3}x^{\frac{3}{2}} + c\right)' = 6x^2 - \frac{2}{3}\frac{3}{2}x^{\frac{3}{2}-1} + 0 = 6x^2 - x^{\frac{1}{2}}$.

- $\int \frac{4x}{x^2+2}dx = 2\ln(x^2 + 2) + c$.

  Check: $\left(2\ln(x^2 + 2) + c\right) = 2\ln'(x^2 + 2) \cdot (x^2 + 2)' = 2\frac{1}{x^2+2} \cdot 2x = \frac{4x}{x^2+2}$.

- $\int(x^3 + 3^x)dx = \frac{1}{4}x^4 + \frac{1}{\ln 3}3^x + c$

- $\int \cot\, x\, dx = \int \frac{\cos x}{\sin x}dx = \ln|\sin x| + c$

In order to find indefinite integrals, it is essential to be confident in differentiation. So we can "guess" the primitive function $F(x)$ of the given function, that means we know its indefinite integral, because $\int f(x)dx = F(x) + c \quad \text{if } F'(x) = f(x)$.

It is also important for finding the indefinite integral to know some rules of integration. Mathematicians need to know more rules; but is is useful for us, too, to learn a few basic rules of integration in order to pass the exam.

Indefinite integral of basic functions

The function f	$\int f(x)dx$	Restrictions
$y=0$	c	$-$
$y=1$	$x+c$	$x \in \mathbb{R}$
$y=x^r$	$\frac{x^{r+1}}{r+1}+c$	$x \in \mathbb{R}, \quad r \in \mathbb{R} \setminus \{1\}$
$y=\frac{1}{x}$	$\ln\lvert x\rvert+c$	$x \neq 0$
$y=a^x$	$\frac{a^x}{\ln a}+c$	$x \in \mathbb{R}, a \in \mathbb{R}, a>0, a \neq 1$
spec. $y=e^x$	e^x+c	$x \in \mathbb{R}$
$y=\sin x$	$-\cos x+c$	$x \in \mathbb{R}$
$y=\cos x$	$\sin x+c$	$x \in \mathbb{R}$
$y=\frac{1}{\cos^2 x}$	$\tan x+c$	$x \neq \frac{\pi}{2}+k\pi$
$y=\frac{1}{\sin^2 x}$	$-\cot x+c$	$x \neq k\pi$

Apart from the content of this table, we can use the table of basic derivatives from the last lesson. Applying it "backwards" we gain further elementary integration formulas.

Two useful rules of integration (theorems) and examples for their use:

Theorem 1:

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c,$$

if $f(x) \neq 0$.

Examples:

- $\int \frac{4x}{x^2-1}dx = 2\int \frac{2x}{x^2-1}dx = 2\ln|x^2-1| + c$
- $\int \frac{x^2}{1-x^3}dx = \frac{1}{-3}\int \frac{-3x^2}{1-x^3}dx = -\frac{1}{3}\ln|1-x^3| + c$

Theorem 2:

$$\int (f(x))^r \cdot f'(x)dx = \frac{(f(x))^{r+1}}{r+1} + c,$$

where $r \in \mathbb{R}$, but $r \neq -1$.

Examples:

- $\int \sqrt[3]{4+2x}dx = \frac{1}{2}\int (4+2x)^{\frac{1}{3}} \cdot 2dx = \frac{1}{2}\frac{(4+2x)^{\frac{1}{3}+1}}{\frac{1}{3}+1} + c = \frac{1}{2}\frac{(4+2x)^{\frac{4}{3}}}{\frac{4}{3}} + c$
- $\int \frac{3x}{\sqrt{7x^2+1}}dx = \frac{3}{14}\int (7x^2+1)^{-\frac{1}{2}} \cdot 14x\,dx = \frac{3}{14}\frac{(7x^2+1)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c =$
  $\frac{3}{14}\frac{(7x^2+1)^{\frac{1}{2}}}{\frac{1}{2}} + c$

- The case of $r=-1$:
  $\int \frac{x}{2x^2-5}dx = \frac{1}{4}\int (2x^2-5)^{-1}\cdot 4x\,dx = \frac{1}{4}\frac{(2x^2-5)^{-1+1}}{-1+1}+c$?? It makes no sense, as it is not defined! In the theorem we are trying to apply, there is $r\neq -1$! (At the solution of this exercise, we have to use Theorem 1.)

Check whether the above solutions are correct!

Exercises:

Find the following indefinite integrals.

1) $\int \sqrt{5x+1}\,dx$

2) $\int \sqrt[3]{3+4x}\,dx$

3) $\int \frac{3}{\sqrt{7x+1}}dx$

4) $\int \frac{x}{x^2+1}dx$

5) $\int \frac{x^2}{1-x^3}dx$

6) $\int x\sqrt{x^2+5}\,dx$

7) $\int \sqrt[3]{6x-3}\,dx$

8) $\int (3-2x)^7 dx$

9) $\int \frac{1}{\sqrt[3]{2-5x}}dx$

10) $\int \frac{2x-3}{x^2-3x+5}dx$

11) $\int \frac{2x^3}{5-x^4}dx$

12) $\int \frac{5}{7+x}dx$

13) $\int \frac{1-\sin^3 x}{\sin^2 x}dx$

14) $\int \frac{3\cdot 2^x-2\cdot 3^x}{2^x}dx$

15) $\int \frac{1}{\tan^2 x}dx$

16) $\int \frac{1}{\sin^2 x\cdot\cos^2 x}dx$

17) $\int \frac{1}{1+9x^2}dx$

Calculate the following definite integrals.

18) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan x\,dx$

19) $\int_{\frac{2\pi}{3}}^{\frac{4\pi}{2}} \sin x\,dx$

20) $\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\,dx$

Note: If you are not very confident using radian values, you can use degrees: 180° instead of π, 90° instead of $\frac{\pi}{2}$, etc.

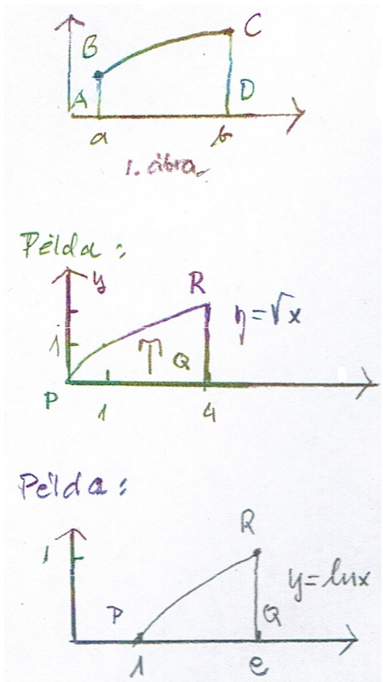
4.1 Applications of the definite integral

In mathematical or engineering books, we encounter formulas for the application of definite integrals very often. Knowing these formulas is not enough to find the solution for practical problems. We have to learn integration properly.

We are going to look through a few topics from the field of geometry. We will also give a formula for finding the centre of mass and prove it.

1. Formulas for geometric applications:

   - The area of the quadrangle $ABCD$ (see in the 1st image), if the curve BC is given as $\{(x,y):\ \ y=f(x),\ \ a\leq x\leq b\}$, can be calculated as follows.



The area under the curve of the function $f(x)$ over the interval $[a, b]$ is

$$T_{ABCD} = \int_a^b f(x)\,dx.$$

Example:

$T_{PQR} = \int_0^4 \sqrt{x}\,dx = \int_0^4 x^{\frac{1}{2}}\,dx = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^4 = \frac{2}{3}\cdot 4^{\frac{3}{2}} - \frac{2}{3}\cdot 0^{\frac{3}{2}} = \frac{2}{3}\left(\sqrt{4}\right)^3 = \frac{2}{3}\cdot 2^3 = \frac{16}{3}.$

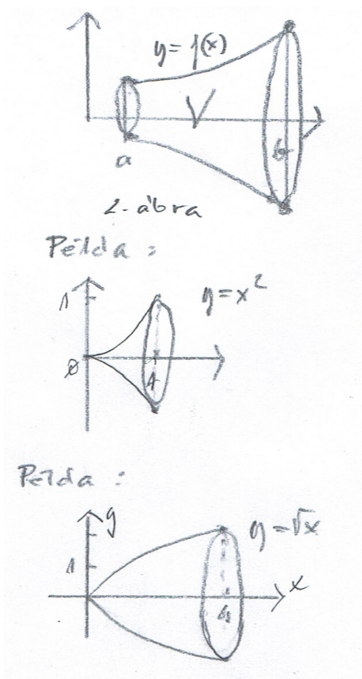
Example:

$T_{PQR} = \int_1^e \ln x\,dx = ?$

For solving the task of finding $\int \ln x\,dx$ we have to know a method that we haven't discussed, so we are not able to calculate the area[1].

[1] We can solve it using *integration by parts:*
$\int \ln x\,dx = \int 1\cdot \ln x\,dx = x\ln x - \int x\cdot\frac{1}{x}\,dx = x\ln x - \int 1\,dx = x\ln x - x + c.$

- Calculating the volume of the body gained by rotating the curve of the function $y = f(x)$ about the x axis (see 2nd image):



$$V = \pi \int_a^b y^2 \, dx = \pi \int_a^b (f(x))^2 \, dx.$$

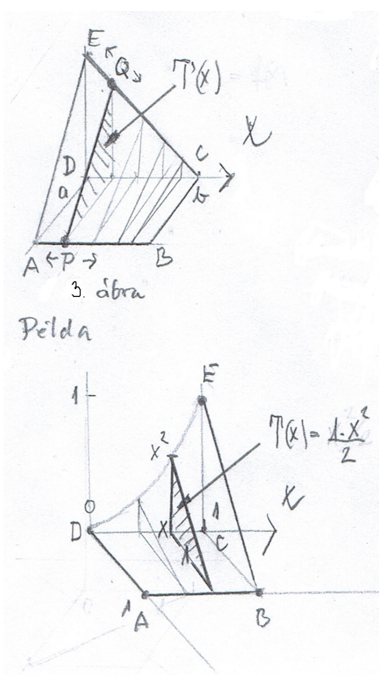
Example:

$V = \pi \int_0^1 y^2 \, dx = \pi \int_a^b \left(x^2\right)^2 dx =$
$\pi \left[\frac{1}{5}x^5\right]_0^1 = \frac{1}{5}\pi.$

Example:

$V = \pi \int_0^4 \left(\sqrt{x}\right)^2 dx = \pi \int_0^4 x \, dx =$
$\pi \left[\frac{1}{2}x^2\right]_0^4 = \frac{\pi}{2} \cdot 4^2 = 8\pi.$

- Calculating the volume of a body, when the area T, gained by intersecting the body with a plane perpendicular to the x axis, is given as a function of x (see the 3rd image):



(We gain the surface $ABCE$ by sliding the segment $\overline{PQ}$ along the line segments AB and CE.)

$V_{ABCDE} = \int_a^b T(x) \, dx.$

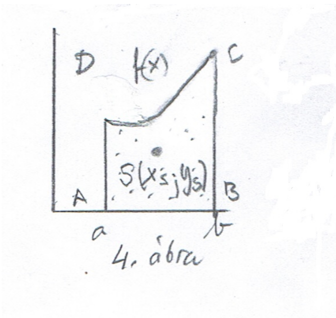
Example:

Calculation of the volume (the "inner" space) of the body $ABCDE$:

$V_{ABCDE} = \int_0^1 \frac{1 \cdot x^2}{2} \, dx =$
$\left[\frac{1}{2} \cdot \frac{1}{3}x^3\right]_0^1 = \frac{1}{6}.$

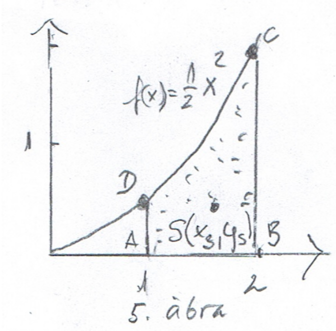
Note: Based on the same formula, we can also calculate the volume of a body gained by rotating a curve about the x axis.

- Coordinates of the centre of mass (S) of a homogeneous "curved trapezoid" (the area under the curve) that you can see in the 4th image:



$$x_s = \frac{\int_a^b x f(x)\, dx}{\int_a^b f(x)\, dx}; \quad y_s = \frac{\frac{1}{2}\int_a^b f^2(x)\, dx}{\int_a^b f(x)\, dx}.$$

($y = f(x)$ is the equation of the curve DC, and $\int_a^b f(x)\, dx$ is the area of the curved trapezoid $ABCD$.)



Example: Find the coordinates of the centre of mass of the homogeneous plate $ABCD$ with $y = \frac{1}{2}x^2$ (see the 5th image):

$$x_s = \frac{\int_1^2 x \cdot \frac{1}{2}x^2\, dx}{\int_1^2 \frac{1}{2}x^2\, dx}; \quad y_s = \frac{\frac{1}{2}\int_1^2 \left(\frac{1}{2}x^2\right)^2\, dx}{\int_1^2 \frac{1}{2}x^2\, dx}.$$

Please do the calculation of the above integrals and find x_s, y_s yourself!

Please make an estimation before doing the calculations, and after finding the exact coordinates x_s and y_s, you will see what is the "error" of your estimation.

In the previous example we worked using the given formulas for the centre of mass. But how can we deduce these formulas?

If we had a live lecture, I would write the following on the board:

Application of the definite integral – calculating the centre of mass.

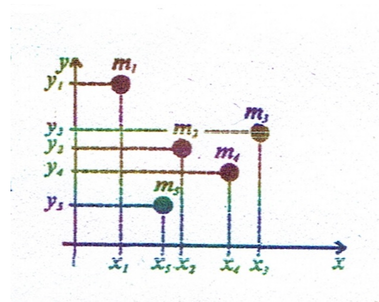
a) The concept of *the centre of mass* is the following (in the plane):

- The torque of a system of masses on the y axis is $m_1x_1 + m_2x_2 + ... + m_nx_n$, so

$$\sum_{i=1}^{n} m_i x_i = x_s \left(\sum_{i=1}^{n} m_i \right);$$

- the torque of the same system on the x axis is $m_1y_1 + m_2y_2 + ... + m_ny_n$, so

$$\sum_{i=1}^{n} m_i y_i = y_s \left(\sum_{i=1}^{n} m_i \right).$$



(As if putting all the mass in the centre.)

- As a conclusion of the above equations, we gain expressions for the coordinates of the centre of mass of the mass system $m_1, m_2, ..., m_n$:

$$x_s = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \quad \text{and} \quad y_s = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}.$$

b) Finding the centre of mass of a plate, bordered by the graph of the function $y = f(x)$.

Let the function $f(x)$ be continuous on the interval $[a, b]$.

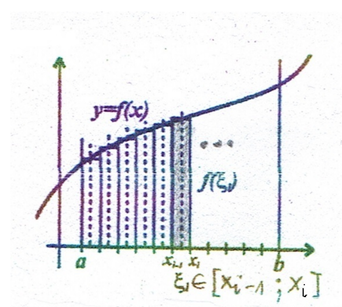
Example: Find the centre of mass of a plate of unit thickness and unit density, bordered by the lines $y = f(x)$, $x = a$, $x = b$ and $y = 0$.

- After dividing the interval $[a, b]$ to n parts, in all of the line segments gained by the division, the part of the plate corresponding to this segment will be approximated by a rectangle with sides $(x_i - x_{i-1})$ and $f(\xi_i)$ $(i = 1, 2, ..., n)$.

- The torque of the ith piece of the plate on the y axis:

$$\xi_i \cdot \{f(\xi_i)(x_i - x_{i-1}) \cdot 1 \cdot 1\}$$

(the multipliers are there for the thickness and the density).



- The torque of the ith piece of the plate on the x axis:

$$\left\{\frac{1}{2} f(\xi_i)\right\} \cdot \{f(\xi_i)(x_i - x_{i-1}) \cdot 1 \cdot 1\}.$$

- The torque of the whole plate, consisting of n rectangular pieces, on the y axis:

$$\sum_{i=1}^{n} \xi_i \cdot \{f(\xi_i)(x_i - x_{i-1}) \cdot 1 \cdot 1\}.$$

– The torque of the whole plate, consisting of n rectangular pieces, on the x axis:

$$\sum_{i=1}^{n} \left\{ \frac{1}{2} f(\xi_i) \right\} \cdot \{ f(\xi_i)(x_i - x_{i-1}) \cdot 1 \cdot 1 \} .$$

– Let $S(x_s, y_s)$ be the centre of mass of the plate. The following equations must hold approximately for the coordinates of the centre of mass:

$$\sum_{i=1}^{n} \xi_i f(\xi_i)(x_i - x_{i-1}) \approx x_s \int_a^b f(x)\, dx$$

and

$$\sum_{i=1}^{n} \frac{1}{2} \left(f(\xi_i)\right)^2 (x_i - x_{i-1}) \approx y_s \int_a^b f(x)\, dx.$$

Let n tend to infinity and $\max_i |x_i - x_{i-1}|$ tend to 0. Then the limits of the above sums are integrals, and the following holds (with equality now):

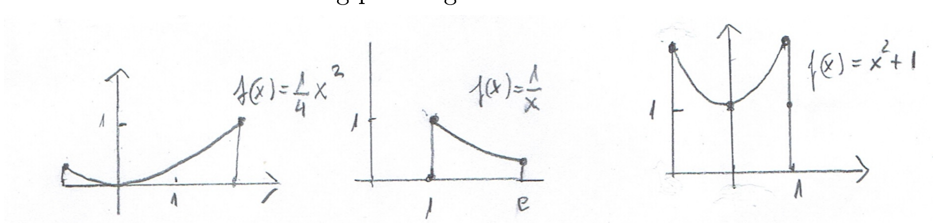
$$\int_a^b x f(x)\, dx = x_s \int_a^b f(x)\, dx \quad \text{and} \quad \int_a^b \frac{1}{2} \left(f(x)\right)^2 dx = y_s \int_a^b f(x)\, dx;$$

so

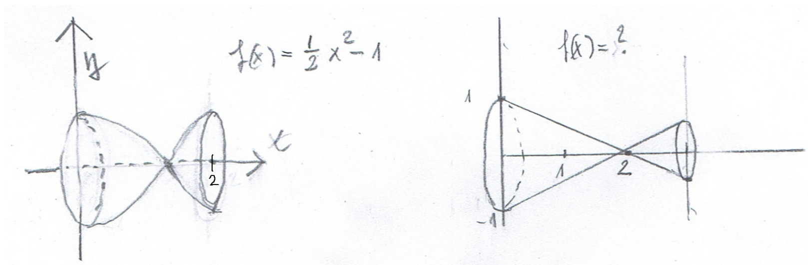
$$x_s = \frac{\int_a^b x f(x)\, dx}{\int_a^b f(x)\, dx} \qquad \text{and} \qquad y_s = \frac{\frac{1}{2} \int_a^b f^2(x)\, dx}{\int_a^b f(x)\, dx}.$$

Exercises

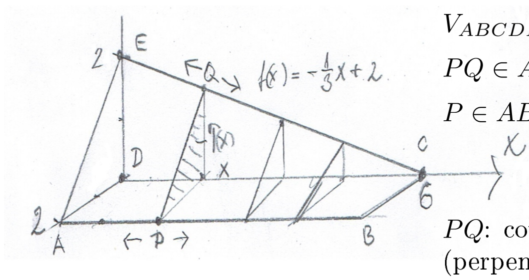
- Find the area of the following plane figures.



- Calculate the volume of the following solids of revolution.



- How much is the volume of the given body in the following 3 cases:

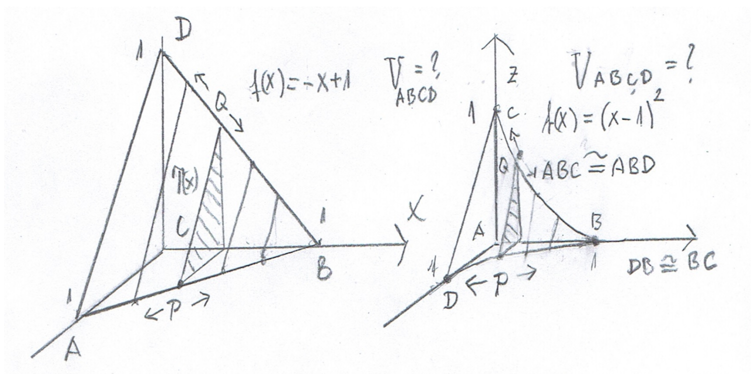


$V_{ABCDE} = ?$

$PQ \in ABCE$ surface,

$P \in AB$; $Q \in EC$

PQ: covering line segments, sliding (perpendicular to the x axis).



Chapter 5

Multivariate function and its derivatives

1. Definition of n dimensional space.

   Let the set S consist of $(x_1, x_2, ..., x_n)$ ordered n-tuples of real numbers.

   The set of ordered tuples of n real numbers we will call **n-dimensional space**, and denote it by $\mathbb{R}^n$.

   a) We can think about the elements of the space $\mathbb{R}^2$, $\mathbb{R}^3$ as location vectors of points $P = (x_1, x_2)$ and $P = (x_1, x_2, x_3)$, respectively.
      We can generalize this to $\mathbb{R}^n$, regarding its elements as location vectors $P = (x_1, x_2, ..., x_n)$.
      The operations that we know in two and three dimensional space, can also be defined for $P = (x_1, x_2, ..., x_n)$ vectors:

      – $\alpha\underline{a} = (\alpha a_1, \alpha a_2, ..., \alpha a_n)$ (multiplication of a vector by a real number);
      – $\underline{a} \pm \underline{b} = (a_1 \pm b_1, a_2 \pm b_2, ..., a_n \pm b_n)$ (addition of vectors);
      – $\underline{a}.\underline{b} = (a_1 b_1 + a_2 b_2 + ... + a_n b_n)$ (scalar product of two vectors).

   b) The **magnitude of a vector** – or length, or absolute value – is defined as

   $$|\underline{a}| = \sqrt{\sum_{i=1}^{n} a_i^2} = \sqrt{a_1^2 + a_2^2 + ... + a_n^2} = \sqrt{\underline{a}.\underline{a}} \ .$$

   The distance of the points A and B in the space $\mathbb{R}^n$, based on the above, is $\overline{AB} = \sqrt{\sum_{i=1}^{n} (a_1 - b_i)^2}$.

c) In the space $\mathbb{R}^n$ the **neighborhood of radius r** about a point P_0 consists of all those points $P \in \mathbb{R}^n$ for which $\overline{PP_0} < r$ holds.

A set S is called *open* if all of its points are *inner points*, that means the set contains a neighborhood of the point P if the point P belongs to the set.

The point P is said to be a *boundary point* of the set S if any neighborhood of P contains a point of S (other than P) and also a point out of S.

The set S is said to be closed if it contains all of its boundary points.

2. The concept of a multivariate function, ways of its definition.

a) The concept of a multivariate function.

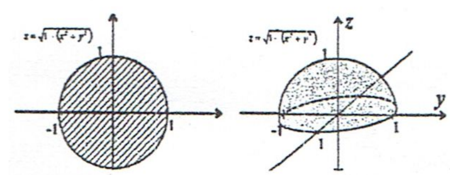
A mapping that assigns a real number to each element (point) of a subset of the space $\mathbb{R}^n$ is called an **n-variate real function**.

Notation:
$(x_1, x_2, ..., x_n) \mapsto f(x_1, x_2, ..., x_n)$, or
$z = f(P) \qquad (P \in \mathbb{R}^n, \;\; z \in \mathbb{R})$.

b) Finding the domain of a function.
$D_f = \{(x, y) \, : \, x \in \mathbb{R}, \; y \in \mathbb{R}\}$ is the set of those points where the function is defined (where it has a value).

Example: find the domain of the following function, and illustrate it as a part of the (x, y) plane.

The domain of the function
$z = f(x, y) = \sqrt{1 - (x^2 + y^2)}$
($\Rightarrow x^2 + y^2 + z^2 = 1 \Rightarrow$ positive half of a sphere):
$D_f = \big\{(x, y) \, : \, x \in \mathbb{R}, \; y \in \mathbb{R}, \; x^2 + y^2 \leq 1\big\}$.



3. Depiction of a function of two variables.

- Depiction of $z = f(x, y) = x^2 + y^2$:

We consider the sections of the graph of the function (which is a surface) with planes parallel to one of the coordinate planes, and conclude what the 3-dimensional image looks like: make an axonometric image.

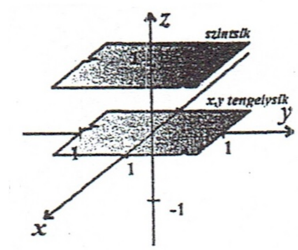
We use the $z = c \;\; (c \in \mathbb{R})$ planes in general, that are parallel to the (x, y) plane.

The intersection with these $z = c$ planes:

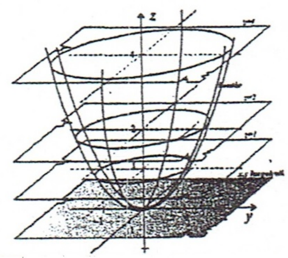
$z = x^2 + y^2$

$z = c \qquad \Rightarrow$

$c = x^2 + y^2 \; (c \geq 0)$



The intersections parallel to the (x, y)-plane are circles, or points in the singular case, but this intersection is not empty only if $c \geq 0$. That means this surface has no points under the (x, y)-plane (z can not be negative).



Examples for intersections:

$$\begin{array}{llllll} z = x^2 + y^2 & & z = x^2 + y^2 & & & \\ \underline{z = 1} & \text{ill.} & \underline{z = 2} & \text{ill.} & \underline{z = 4} & \Rightarrow \\ 1 = x^2 + y^2 & & 2 = x^2 + y^2 & & 4 = x^2 + y^2 & \end{array}$$

4. Differentiation of functions of two variables.

   a) Partial derivative.

   Let the function $z = f(x, y)$ be defined in a neighborhood of P_0.
   If the limit $\lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$ exists, then we say $f(x, y)$ is partially differentiable at P_0 with respect to x, and its derivative at P_0 is denoted by the symbol $|_{\frac{\partial f}{\partial x}} P_0$, $f'_x(P_0)$ or $z'_x(x_0, y_0)$.

   The partial derivative with respect to x at P_0: $|_{\frac{\partial f}{\partial x}} P_0 = \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$

   The partial derivative with respect to x: $\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$

   The partial derivative with respect to y at P_0: $|_{\frac{\partial f}{\partial y}} P_0 = \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}$

   The partial derivative with respect to y: $\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$

   Example 1.

   $f(x, y) = 7x^3y^2 + 2\sqrt{x} - 3y^4 + 5 \quad P_0(1; 0)$

   $\frac{\partial f}{\partial x} = 7 \cdot 3x^2 \cdot y^2 + 2 \cdot \frac{1}{2}x^{-\frac{1}{2}} - 0 + 0$

   $|_{\frac{\partial f}{\partial x}} P_0(1; 0) = 7 \cdot 3 \cdot 1^2 \cdot 0^2 + 2 \cdot \frac{1}{2} \cdot 1^{-\frac{1}{2}} = 1$

   $\frac{\partial f}{\partial y} = 7x^3 \cdot 2y + 0 - 3 \cdot 4y^3 + 0$

   $|_{\frac{\partial f}{\partial y}} P_0(1; 0) = 7 \cdot 1^3 \cdot 2 \cdot 0 - 3 \cdot 4 \cdot 0^3 + 0 = 0$

   Example 2.

   $z = f(x, y) = \sqrt{x^3y^2}; \quad P_0(1; 2)$

$|_{\frac{\partial f}{\partial x}} P_0 = ?$; $|_{\frac{\partial f}{\partial y}} P_0 = ?$

$|_{\frac{\partial f}{\partial x}} P_0 = |_{\frac{\partial z}{\partial x}} P_0 = |_{z'_x} P_0 = |_{\frac{\partial \sqrt{x^3y^2}}{\partial x}} P_0 = |_{\frac{\partial (x^3y^2)^{1/2}}{\partial x}} P_0 =$
$|_{\left(\frac{1}{2}(x^3y^2)^{-\frac{1}{2}} \cdot 3x^2y^2\right)} P_0(1,2) =$

$= \frac{1}{2}(1^3 2^2)^{-\frac{1}{2}} \cdot 3 \cdot 1^2 \cdot 2^2 = 3$

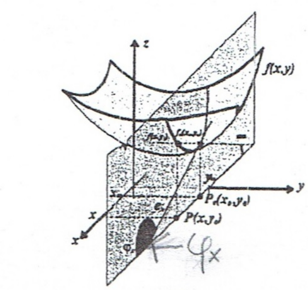
$|_{\frac{\partial f}{\partial y}} P_0 = |_{\frac{\partial (x^3y^2)^{1/2}}{\partial y}} P_0 = |_{\left(\frac{1}{2}(x^3y^2)^{-\frac{1}{2}} \cdot x^3 \cdot 2y\right)} P_0(1,2) =$
$\frac{1}{2}(1^3 2^2)^{-\frac{1}{2}} \cdot 1^3 \cdot 2 \cdot 2 = 1$

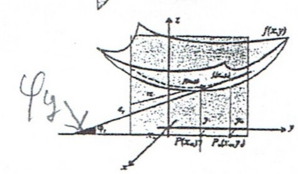
Note: $\sqrt{x^3y^2} = (x^3y^2)^{1/2}$ is a composite function. The outside function is the square root, the inside function is x^3y^2.

b) The geometric meaning of the partial derivative.

The geometric meaning of the partial derivative of the function $f(x,y)$ at P_0:

- $\tan(\phi_x)$ is the slope of the tangent line e_x,
  $|_{\frac{\partial f}{\partial x}} P_0 = \lim_{x \to x_0} \frac{f(x,y_0) - f(x_0,y_0)}{x - x_0} = \tan(\phi_x)$;
- $\tan(\phi_y)$ is the slope of the tangent line e_y.
  $|_{\frac{\partial f}{\partial y}} P_0 = \lim_{y \to y_0} \frac{f(x_0,y) - f(x_0,y_0)}{y - y_0} = \tan(\phi_y)$.





The values $|_{\frac{\partial f}{\partial x}} P_0$ and $|_{\frac{\partial f}{\partial y}} P_0$ describe "the slope of the function" along two directions $-$ parallel to the x- and the y-axis $-$, at $P_0(x_0, y_0)$.

Example: find the slope of the tangent lines at the point x_0, y_0 drawn to the curves gained by intersecting the following surfaces by planes parallel to the (x,z) and the (y,z) coordinate planes, going through the point $x_0, y_0, f(x_0,y_0)$.

$f(x,y) = z = x^2y^3 \qquad x_0 = 1 \qquad y_0 = 2$

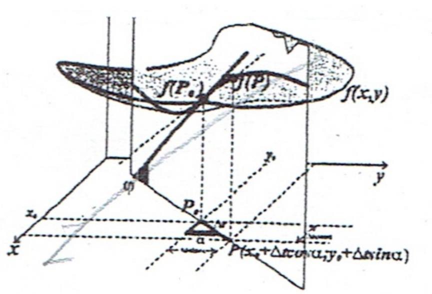
$|_{\frac{\partial f}{\partial x}} P_0 = |_{\frac{\partial (x^2y^3)}{\partial x}} P_0(1,2) = |_{2xy^3} P_0(1,2) = 2 \cdot 1 \cdot 2^3 = 2^4 = \tan \phi_x$

is the slope of the tangent line of the curve (that is the graph of a function of one variable) gained by intersecting the $z = x^2y^3$ surface at $(x_0, y_0) = (1,2)$ with a plane parallel to the (x,z) coordinate plane.

(For the slope of the intersection parallel with the (y,z) plane in the given point $|_{\frac{\partial f}{\partial y}} P_0$ yields the result!)

c) Concept of directional derivative: The slope of the tangent line at the point $(P_0; f(P_0))$ to the curve gained by intersecting the $z = f(x, y)$ surface with a plane perpendicular to the (x, y)-plane, going through the point P_0 and enclosing an angle α with the x axis.

The value of the directional derivative:
$f'_\alpha(P_0) = f'_\alpha(x_0, y_0) =$
$|_{\frac{\partial f}{\partial x}} P_0 \cdot \cos\alpha + |_{\frac{\partial f}{\partial y}} P_0 \cdot \sin\alpha.$
Example: $f(x, y) =$
$4x^2y^3 \quad P_0(1; 1) \quad \alpha = \frac{\pi}{3}$
$f'_{\frac{\pi}{3}}(1; 1) =$
$|_{8xy^3} P_0 \cdot \cos\frac{\pi}{3} + |_{12x^2y^2} P_0 \cdot \sin\frac{\pi}{3} =$
$8 \cdot \frac{1}{2} + 12 \cdot \frac{\sqrt{3}}{2} = 14,39.$

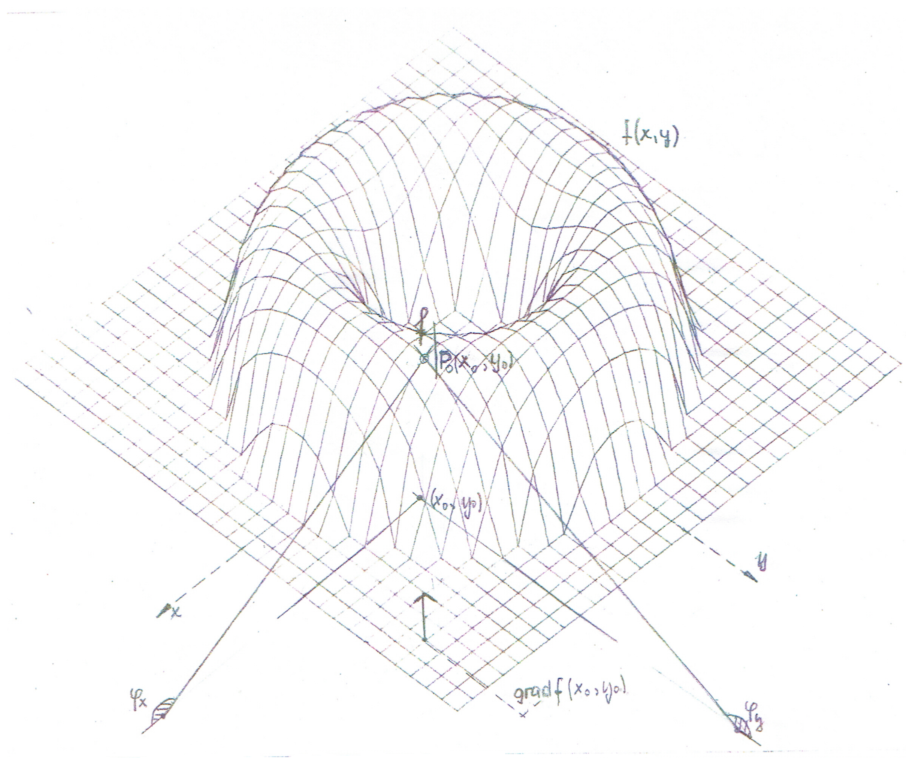


d) Definition of the gradient of a function of two variables at P_0:
The gradient of a function $f(x, y)$ at P_0 is defined as

$$|_{grad\, f} P_0 = |_{\frac{\partial f}{\partial x}} P_0 \underline{i} + |_{\frac{\partial f}{\partial y}} P_0 \underline{j}.$$

The gradient is vector (depending on P_0) in the (x, y)-plane, with the following important property: starting from P_0, changing x, y along this vector, we get the biggest increase in the function compared to other directions; we gain the biggest growth in this direction. (As the measure of the growth is the directional derivative in this direction, that is the scalar product of the gradient vector with the same direction unit vector, so the magnitude of the growth, the derivative, is equal to the magnitude of the gradient vector.)

Example for calculating the gradient:
$f(x, y) = 4x^2y^3 \qquad P_0(1; 1)$
$grad|_f P_0 = |_{(8xy^3\underline{i} + 12x^2y^2\underline{j})} P_0 = 8\underline{i} + 12\underline{j}.$



5.1 Exercises

Find the domain of the following functions and illustrate it in the (x, y)-plane.

- $z = \frac{1}{xy}$
- $z = \ln(x - 2y)$
- $z = xy(\sqrt{x} + \sqrt{y})$
- $z = \sqrt{xy}\sin(x + y)$
- $z = \sqrt{x - y^2}$
- $z = \arcsin(x^2 + 4y^2)$
- $z = \sqrt{1 - x^2} + \sqrt{1 - y^2}$
- $z = \frac{1}{\sqrt{x+y}} + \frac{1}{\sqrt{x-y}}$
- $z = \frac{1}{\sin x} + \frac{1}{\cos y}$

Differentiation

Find the first partial derivatives of the following functions.

- $z = 2x^3 - 5x^2y + 3xy^2 - 8y^2 + 7xy + 6x$
- $z = \ln x^y$
- $xe^y + ye^x$
- $z = \frac{x}{\sqrt{x^2+y^2}}$
- $2xy + ye^{\sqrt{3-x}}$
- $\cot \frac{x}{y}$
- $z = \ln \tan \frac{x}{y}$
- $z = \tan \frac{x+y}{1-xy}$

Calculate the slope of the tangent line to the intersection curves in the following surfaces at the point belonging to $(x_0; y_0)$, parallel to (x, z) and (y, z) coordinate planes, respectively.

- $z = x^2y \qquad x_0 = 1 \qquad y_0 = 2$
- $z = \sin^2 x - y^2 \qquad x_0 = \frac{\pi}{3} \qquad y_0 = 1$
- $x^2 + y^2 + z^2 = 4 \qquad x_0 = 0,5 \qquad y_0 = 1$
- $z = \sqrt{4x^2 + 8y^2} \qquad x_0 = 1 \qquad y_0 = 2$

Find the directional derivative of the following function at the given point, in the given direction.

- $z = x^2 + 3y^2 \qquad P_0(-2; 1) \qquad \alpha = 30°$
- $z = x^y \qquad P_0(2; 3) \qquad \alpha = \frac{\pi}{4}$
- $z = xy - \frac{3}{xy} \qquad P_0(4; -2) \qquad \alpha = 120°$
- $z = \sqrt{x^2 + y^2} \qquad P_0(3; 4) \qquad \alpha = \frac{7\pi}{6}$
- $z = x^2 - y^2 \qquad P_0(1; 1) \qquad \alpha = 60°, \quad \alpha = 240°$

Find the gradient of the following functions at the given point.

- $z = 6xy \qquad P_0(3; 5)$
- $z = 4x^2 + 3y^2 \qquad P_0(1; 7)$
- $z = \sin x \cos y \qquad P_0(0; \frac{\pi}{3})$
- $z = \frac{1}{\sqrt{x^2+y^2}} \qquad P_0(1; 1)$

Illustrate the domain of integration of the following double integrals.

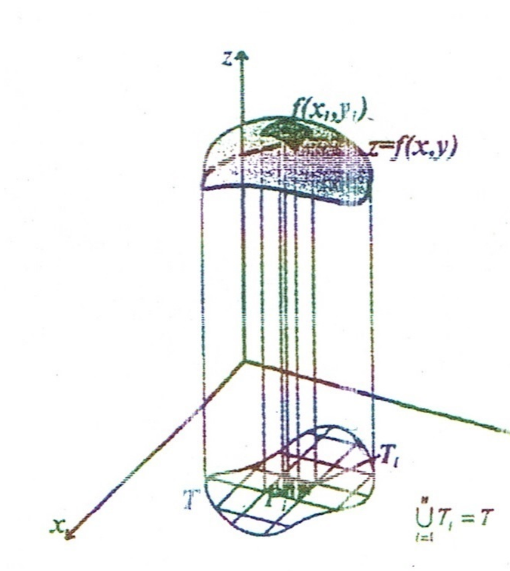
- $\int_0^2 \int_{y-1}^{y+2} f(x,y)\,dx\,dy$
- $\int_{-2}^{2} \int_{1-\frac{x^2}{8}}^{2+\frac{x^2}{4}} f(x,y)\,dy\,dx$
- $\int_0^4 \int_0^{4x-x^2} f(x,y)\,dy\,dx$
- $\int_{-\sqrt{3}}^{\sqrt{3}} \int_1^{\sqrt{4-y^2}} f(x,y)\,dx\,dy$
- $\int_0^1 \int_{-x}^{x^2} f(x,y)\,dy\,dx$
- $\int_0^1 \int_{-\sqrt{1-y^2}}^{1-y} f(x,y)\,dx\,dy$
- $\int_0^2 \int_y^{3-\frac{y}{2}} f(x,y)\,dx\,dy$
- $\int_0^1 \int_y^{\sqrt{y}} f(x,y)\,dx\,dy$

Calculate the following double integrals.

- $\int_0^1 \int_y^1 (x^2+y^2)\,dx\,dy$
- $\int_0^3 \int_0^{3+y} e^{2x+3y}\,dx\,dy$
- $\int_1^2 \int_{\frac{1}{x}}^{x} \frac{x^2}{y^2}\,dy\,dx$
- $\int_0^1 \int_{y^2}^{\sqrt{y}} (x^2+y)\,dx\,dy$
- $\int_1^3 \int_2^5 (5x^2y-2y^3)\,dx\,dy$

5.2 Integrating two-variable functions

The double integral of a function $z = f(x,y)$ over some area (domain) T - denoted by $\int_T f(x,y)\,dT$ or $\iint_T f(x,y)\,dT$ - is defined as the result of the following approximation procedure:



- The area (domain) T is arbitrarily divided into "n" smaller areas (elementary domains).
- Inside (or on the border) of each elementary domain we consider an arbitrary point $P_i(x_i, y_i)$.
- The value of the function $z = f(x,y)$ at the point $P_i(x_i, y_i)$ - that is $f(x_i, y_i)$ - will be multiplied by the area of the corresponding elementary domain: T_i, resulting in the product $f(x_i, y_i)T_i$.
- The products $f(x_i, y_i)T_i$ will be summed up, yielding the sum $\sum_{i=1}^{n} f(x_i, y_i)T_i$.
  This is called the integral-approximating sum of

the two-variable function over the domain T.

The same way, we can produce a sequence of integral-approximating sums.

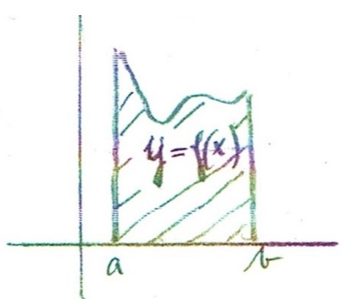
- Take the limit of the previously given sums with the conditions $n \to \infty$ and $\max_i |T_i| \to 0$, that is

$$\lim_{\substack{n \to \infty \\ \max_i |T_i| \to 0}} \sum_{i=1}^{n} f(\xi_i, y_i) T_i.$$

If the above limit exists, and neither does it depend on the division of the domain T into elementary domains, nor on the choice of the points $P_i(x_i, y_i)$, then we call it the integral of the function $z = f(x, y)$ over the domain T - or alternatively: its double integral. That is

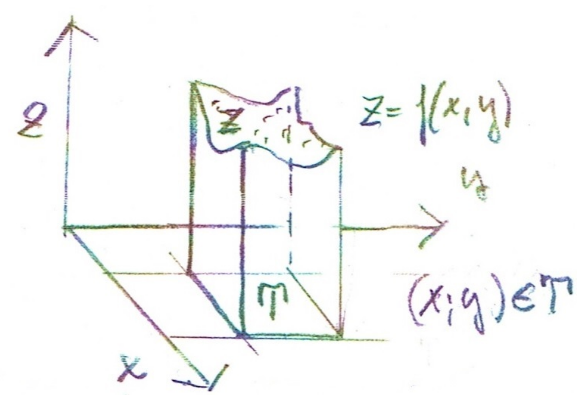
$$\int_T f(x, y)\, dT = \iint_T f(x, y)\, dT = \lim_{\substack{n \to \infty \\ \max_i |T_i| \to 0}} \sum_{i=1}^{n} f(\xi_i, y_i) T_i.$$

If we compare the meaning of the integral $\int_a^b f(x)\, dx$ defined over the interval $[a, b]$ and the integral $\iint_T f(x, y)\, dT$ defined over the domain T, the similarity is obvious:



$\int_a^b f(x)\, dx$ yields an area

(if $f(x) \geq 0 \quad x \in [a,\, b]$)

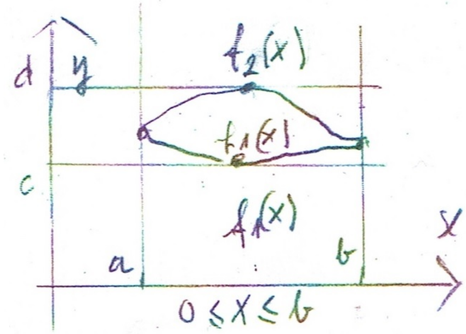


$\iint_T f(x, y)\, dT$ yields a volume (if $f(x, y) \geq 0$ over T)

Calculation of the integral of a two-variable function (double integral).
The value of $\iint_T f(x, y)\, dT$:

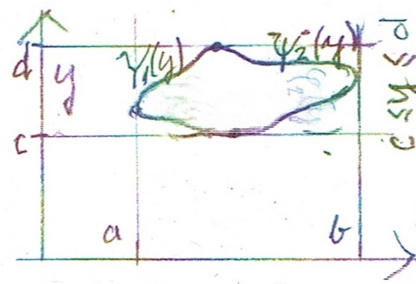
$$\iint_T f(x,y)\,dT =$$

$$= \int_a^b \left(\int_{f_1(x)}^{f_2(x)} f(x,y)\,dy \right) dx$$



$$\iint_T f(x,y)\,dT =$$

$$= \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx \right) dy$$



On the right hand side, there are two consecutive (embedded) integrals to solve.

Examples:

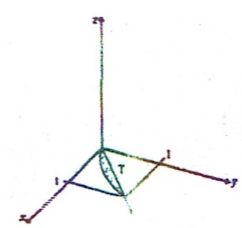
- Calculate the integral of the function $z = f(x,y) = x^2y$ over the domain T (see the below images for T, the domain of integration).

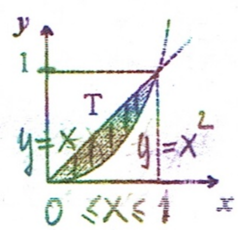
  – First solution.

    The set of points belonging to the domain T can be described by the following two conditions (considering y as a function of x):

    $$T = \left\{ \begin{array}{c} 0 \le x \le 1 \\ x^2 \le y \le x \end{array} \right\}$$

    Using these conditions the double integral over the domain T can be written as:





$$\iint_T x^2y\,dT = \int_0^1 \int_{x^2}^{x} x^2y\,dy\,dx = \int_0^1 \left[x^2 \cdot \frac{y^2}{2} \right]_{y=x^2}^{y=x} dx = \int_0^1 \left(x^2 \cdot \frac{x^2}{2} - x^2 \cdot \frac{x^4}{2} \right) dx =$$

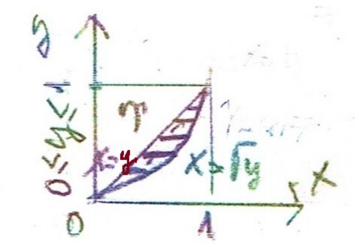
$$= \int_0^1 \left(\frac{1}{2}x^4 - \frac{1}{2}x^6 \right) dx = \left[\frac{x^5}{2\cdot 5} - \frac{x^7}{2\cdot 7} \right]_{x=0}^{x=1} = \left(\frac{1}{10} - \frac{1}{14} \right) - (0) = \frac{2}{70} = \frac{1}{35}.$$

– Second solution.

The domain T can also be described by the following two conditions (considering x as a function of y):

$$T = \left\{ \begin{array}{c} 0 \le y \le 1 \\ y \le x \le \sqrt{y} \end{array} \right\}$$

Using these conditions, the double integral over the domain T can be written as:



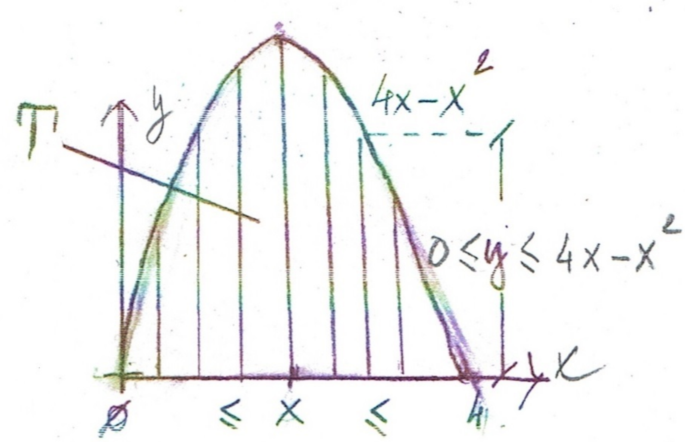
$$\iint_T x^2 y \, dT = \int_0^1 \int_y^{\sqrt{y}} x^2 y \, dx \, dy = \int_0^1 \left[\frac{x^3}{3} \cdot y \right]_{x=y}^{x=\sqrt{y}} dy = \int_0^1 \left(\frac{y^{3/2}}{3} \cdot y - \frac{y^3}{3} \cdot y \right) dx =$$

$$= \int_0^1 \left(\frac{y^{5/2}}{3} - \frac{y^4}{3} \right) dx = \left[\frac{y^{\frac{7}{2}}}{3 \cdot \frac{7}{2}} - \frac{y^5}{3 \cdot 5} \right]_{y=0}^{y=1} = \left(\frac{2}{21} - \frac{1}{15} \right) - (0) = \frac{3}{105} = \frac{1}{35}.$$

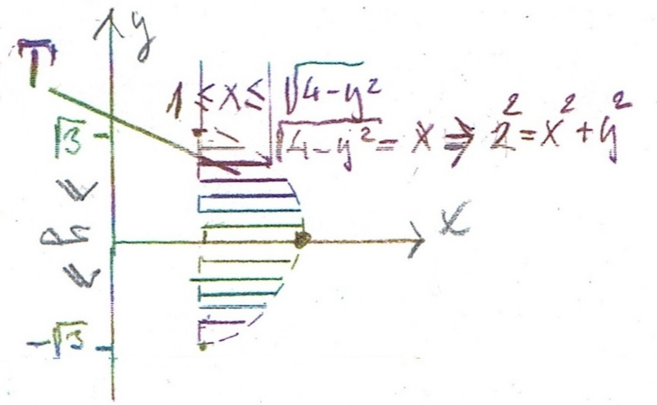
- Depict the domain of the following double integrals.

$$\int_0^4 \int_0^{4x-x^2} f(x,y) \, dy \, dx$$

$$\iint_T f(x,y) \, dy \, dx \qquad T : \left\{ \begin{array}{l} 0 \le x \le 4 \\ 0 \le y \le 4 \end{array} \right.$$



$$\int_{-\sqrt{3}}^{\sqrt{3}} \int_1^{\sqrt{4-y^2}} f(x,y) \, dx \, dy \qquad T : \Big\{$$

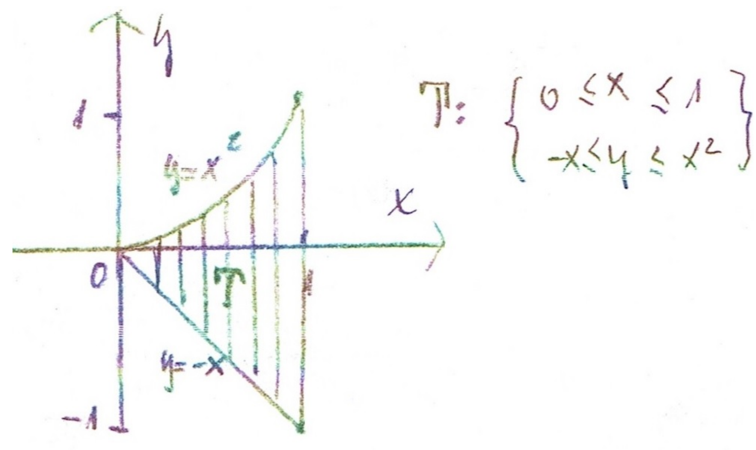


- Depict the domain of the following double integral, and calculate its value.

$$\int_0^1 \int_{-x}^{x^2} x^2 y \, dy \, dx = \iint_T x^2 y \, dT$$

$$c) \quad \int_0^1 \left(\frac{1}{2}x^6 - \frac{1}{2}x^4 \right) dx = \left[\frac{1}{2} \cdot \frac{x^7}{7} - \frac{1}{2} \cdot \frac{x^5}{5} \right]_0^1 = \left(\frac{1}{14} - \frac{1}{10} \right) - (0)$$

$$a) \qquad T : \left\{ \begin{array}{c} 0 \le x \le 1 \\ -x \le y \le x^2 \end{array} \right\}$$



$$b) \quad \int_{-x}^{x^2} x^2 y \, dy = \left[x^2 \cdot \frac{y^2}{2} \right]_{y=-x}^{y=x^2} =$$

$$= x^2 \cdot \frac{\left(x^2\right)^2}{2} - x^2 \cdot \frac{(-x)^2}{2} =$$

$$= \frac{1}{2}x^6 - \frac{1}{2}x^4$$